Quantum Limits on Sensing and Imaging

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Quantum Waveform Sensing

Estimation/detection of classical waveforms $x(r, t)$ using quantum systems

Examples: optical interferometry, optical imaging, optomechanical force sensing (gravitational-wave detection), atomic magnetometry, gyroscopy, etc.
Optical Phase and Frequency Estimation

QCRB [Tsang, Wiseman, and Caves, PRL 106, 090401 (2011); unpublished]:

\[
\phi(t) = \int_{-\infty}^{\infty} dt' h(t - t') x(t'), \quad \langle \delta x^2 \rangle \geq \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{4|h(\omega)|^2 S_{\Delta \hat{f}}(\omega) + 1/S_x(\omega)},
\]

\[
e.g., \quad S_{\Delta \hat{f}}^{\text{coh}}(\omega) = \frac{\bar{P}}{\hbar \omega_0}, \quad S_x^{\text{OU}}(\omega) = \frac{\kappa}{\omega^2 + \epsilon^2}. \quad (1)
\]

Achieved by homodyne phase-locked loop + Smoothing [ Personick IEEE TIT 17, 240 (1971); Tsang, Shapiro, and Lloyd, PRA 78, 053820 (2008); 79, 053843 (2009)]

Wheatley et al., PRL 104, 093601 (2010).

Interferometry, ranging, velocimetry, clock synchronization, coherent comm., etc.
Optomechanical Force Sensing

\[ \langle \delta x^2 \rangle \geq \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(4/\hbar^2)S_{\Delta q}(\omega) + 1/S_x(\omega)} . \]  (2)

Optical Phase Waveform Detection

- Binary hypothesis testing $\mathcal{H}_0 : \phi(t) = 0$, $\mathcal{H}_1 : \phi(t) = x(t)$

- Tsang, unpublished:

$$
\mathcal{H}_1 : \phi = x(t) \\
\mathcal{H}_0 : \phi = 0
$$

- **Photon Counter**

- **Nonzero count**: choose $\mathcal{H}_1$
- **Zero count**: choose $\mathcal{H}_0$

$$
D(-\alpha)
$$

$$
P_e \geq \frac{1}{2} \left( 1 - \sqrt{1 - 4P_0 P_1 F} \right), \quad (3)
$$

$$
F = \int D x P[x] \left| \langle \psi | \exp \left[ i \int_{t_0}^{T} dt \hat{I}(t)x(t) \right] |\psi\rangle \right|^2. \quad (4)
$$

- **Kennedy receiver** has optimal error exponent for stochastic waveform detection with coherent state.

- Homodyne performance depends on **prior waveform statistics**
Optomechanical Force Detection

**Tsang and Nair, arXiv:1204.3697 (2012):**

\[
P_e \geq \frac{1}{2} \left(1 - \sqrt{1 - 4P_0 P_1 F}\right), \quad F = \int Dx P[x] \left| \langle \psi|T \exp \left[i \int_{t_0}^{T} dt \hat{q}_0(t)x(t) \right]|\psi\rangle \right|^2
\]

Because the bound is achievable for deterministic \(x(t)\) and not limited by backaction noise, SQL can definitely be overcome.

QNC + Kennedy receiver achieves optimal error exponent.
Decoherence

- Significant decoherence/loss rules out any significant quantum enhancement using squeezed state/nonclassical state of light (Durkin et al./Escher et al.)
- No-go for nonclassical light in space optics applications
- No result yet about decoherence in waveform estimation/detection, surprise unlikely
- Quantum illumination (Lloyd/Erkmen/Guha/Shapiro/Giovannetti et al.): up to 6 dB improvement in error exponent!
  - Producing squeezed state requires strong pump with way more photons
  - can be achieved by coherent state with 6 dB more photons
  - Known receivers can’t get to 6 dB
  - Low-photon-number regime only
  - Gaussian noise strengths are assumed to be different under hypotheses for QI to be useful, passive target detection may be better in practice
- Quantum Metrology with POVMs
Quantum Imaging

- **Ghost imaging**: Shih/Shapiro/Erkmen
- **Sub-Rayleigh quantum lithography/imaging**: Boto *et al.*, PRL 85, 2733 (2000); Tsang, PRL 102, 253601 (2009); Giovannetti *et al.*, PRA 79, 013827 (2009)
- **Experiments**: D’Angelo *et al.*, PRL 87, 013602 (2001); Guerrieri *et al.*, PRL 105, 163602 (2010); Shin *et al.*, PRL 107, 083603 (2011)
- **Classical computational sub-Rayleigh imaging**: STORM/PALM [Zhuang, Nature Photon. 3, 365 (2009)]; STED (Hell), etc.
- Computational imaging for astronomy [Fienup]
- Not much rigorous work in quantum imaging that uses estimation/detection theory
Quantum Camera Design

- start with multi-spatial-mode $\rho_x(r)$ (e.g., multimode thermal or coherent state)

- Record with imaging system/CCD/interferometer/digital holography (model by POVM $E[y(r)]$)

$$P[y(r)|x(r')] = \text{tr} \left\{ E[y(r)] \Phi_{\text{aperture}} \Phi_{\text{dифффракт}} \rho_x(r') \right\}$$

- Quantum bounds: multiparameter QCRB, etc.

- Do conventional imaging systems saturate these bounds?

- How to implement optimal POVM?
Estimation of coherence:

\[ \Gamma_{ab} = \langle b^\dagger a \rangle, \quad g^{(1)} = \frac{\langle b^\dagger a \rangle}{\sqrt{\langle b^\dagger b \rangle \langle a^\dagger a \rangle}} \] (normalized). (7)
Old-School Quantum Optics

$P$ representation:

$$\rho = \int d^2 \alpha d^2 \beta \Phi(\alpha, \beta) |\alpha, \beta\rangle \langle \alpha, \beta|.$$  \hspace{1cm} (8)

- $\Phi(\alpha, \beta)$ is a two-mode zero-mean Gaussian for thermal light, i.e. no entanglement
- weak thermal light $\epsilon \equiv \langle a^\dagger a \rangle = \langle b^\dagger b \rangle \ll 1$ in photon-number basis:

$$\rho = (1 - \epsilon)|0, 0\rangle \langle 0, 0| + \frac{\epsilon}{2} [ |0, 1\rangle \langle 1, 0| + |1, 0\rangle \langle 1, 0| + g^* |0, 1\rangle \langle 1, 0| + g |1, 0\rangle \langle 0, 1| ] + O(\epsilon^2),$$

$$P(y|g) = \text{tr} [E(y)\rho].$$  \hspace{1cm} (9)

- Classical Fisher information for $g = g_1 + ig_2$:

$$F_{jk} = \left\langle \frac{\partial}{\partial g_j} \ln P \frac{\partial}{\partial g_k} \ln P \right\rangle, \hspace{1cm} \Sigma \geq \frac{1}{M} F^{-1}$$  \hspace{1cm} (11)
Bound for Local Measurements

- Nonlocal measurements (direct detection, shared-entanglement): \( ||F|| \sim \epsilon \).
- A necessary condition for local (LOCC) measurement is the PPT condition applied to the POVM [Terhal et al., PRL 86, 5807 (2001)]. Then \( ||F|| \leq \epsilon^2 + O(\epsilon^3) \).
- Generalizable to repeated LOCC measurements
- Quantum nonlocality in measurement of nature, even if the state has no entanglement.
Misc.

- **Quantum Ziv-Zakai bounds** [Tsang, PRL 108, 230401 (2012)]

- **Continuous quantum hypothesis testing** (for tests of physics using continuous quantum measurements) [Tsang, PRL 108, 170502 (2012)]

- **Cavity quantum microwave photonics** [Tsang, PRA 81, 063837 (2010); 84, 043845 (2011)]