A lecture on

“Quantum Measurements”

at the

Quantum Communication, Sensing and Measurement in Space

workshop of the Keck Institute for Space Studies.

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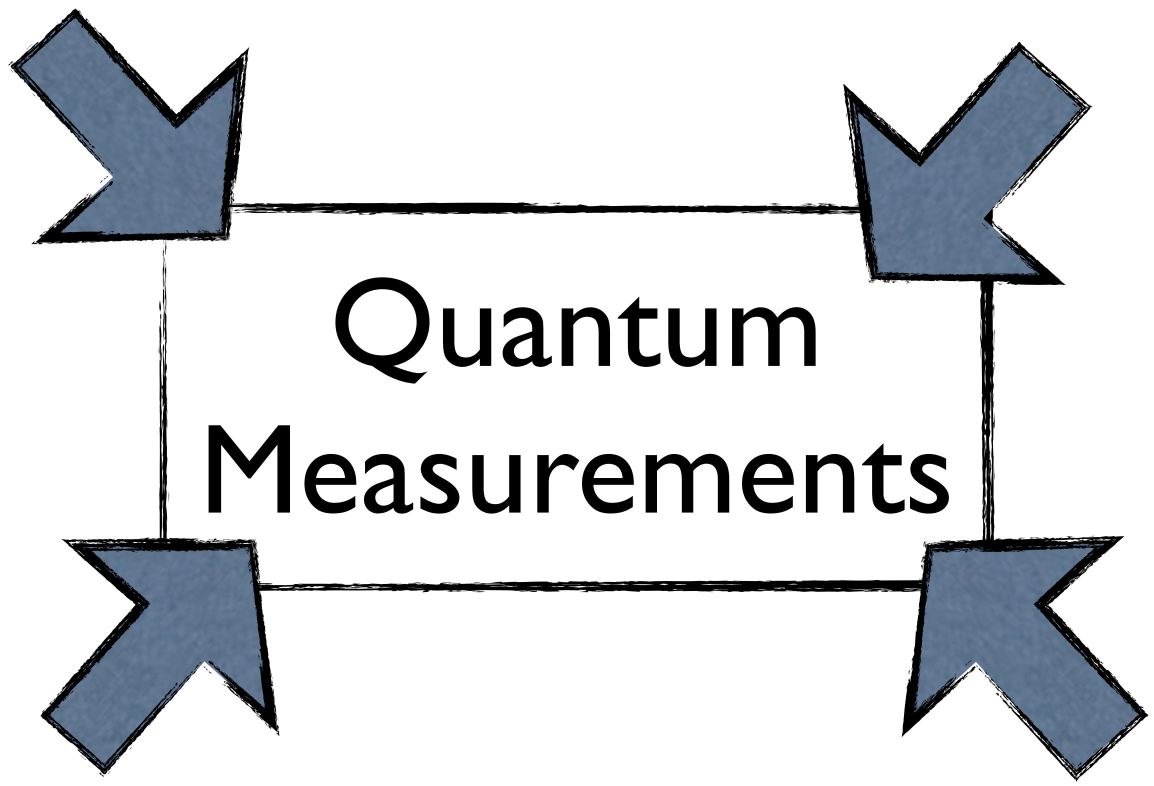
I) Conceptual/ontological aspects

II) Mathematical formalization



III) Practical/technological aspects

IV) Applications

In quantum mechanics a Quantum Measurement (QM) is any process which allows us to acquire (classical) information on the state of a (quantum) system.

“ability of discriminating among different alternatives”

“present condition of a system”

STATES

QUANTUM SYSTEM S $\longleftrightarrow$ HILBERT SPACE (say of finite dimension d)

The physical states of an isolated (closed) quantum system S correspond to the normalized vectors of the space $\mathcal{H}$

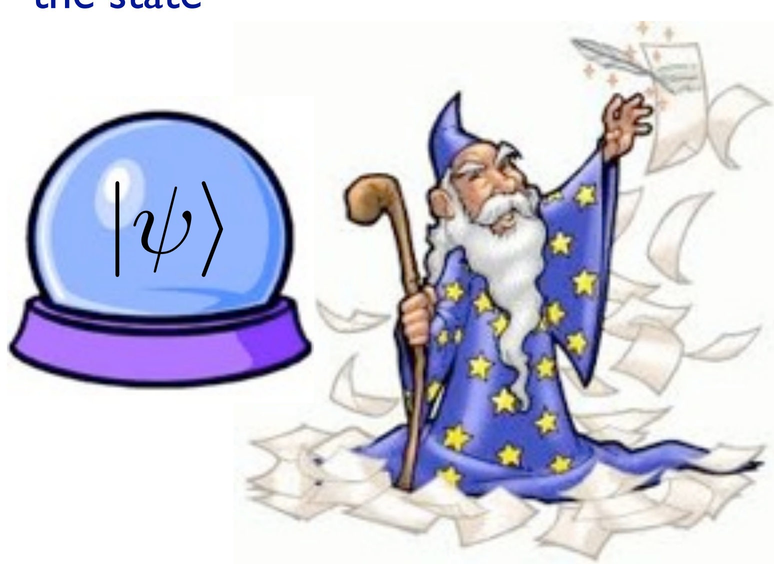
$$|\psi\rangle \in \mathcal{H} \qquad \| |\psi\rangle \| \equiv \sqrt{\langle\psi|\psi\rangle} = 1$$

$|\psi\rangle$ is not a physical quantity nor an observable quantity ...

$|\psi\rangle$ is a mathematical (abstract) object which provides the most precise characterization of the system state: i.e.

it contains in a compact form all the instructions needed for preparing the state

it can be used to predict the outcomes of any measurement performed on the state



STATES

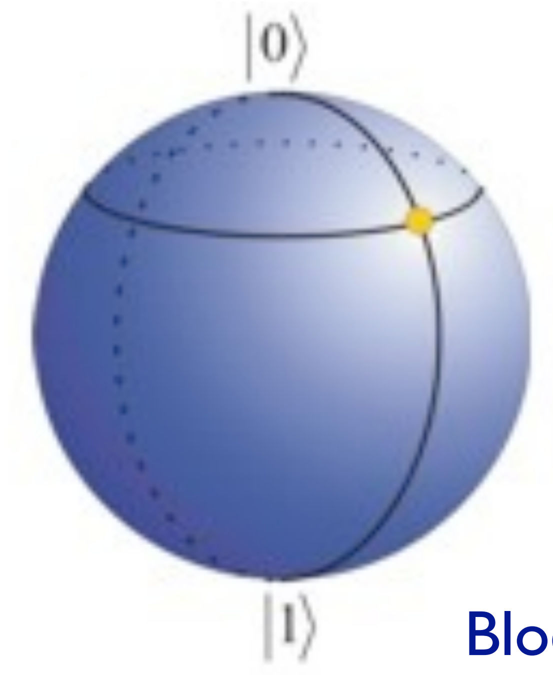
Example: the QUBIT $(\dim \mathcal{H} = d = 2)$

$|0\rangle, |1\rangle \qquad \langle 1|0\rangle = 0$

superposition

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$



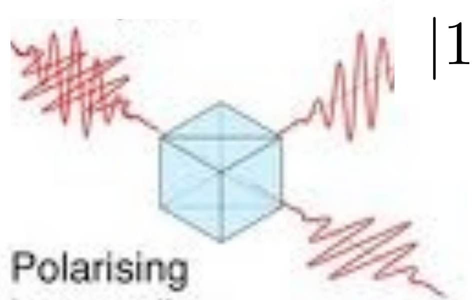
Bloch Sphere

Example: *two level atom*

$|1\rangle = |e\rangle$

$|0\rangle = |g\rangle$

Example: *polarized single photon*



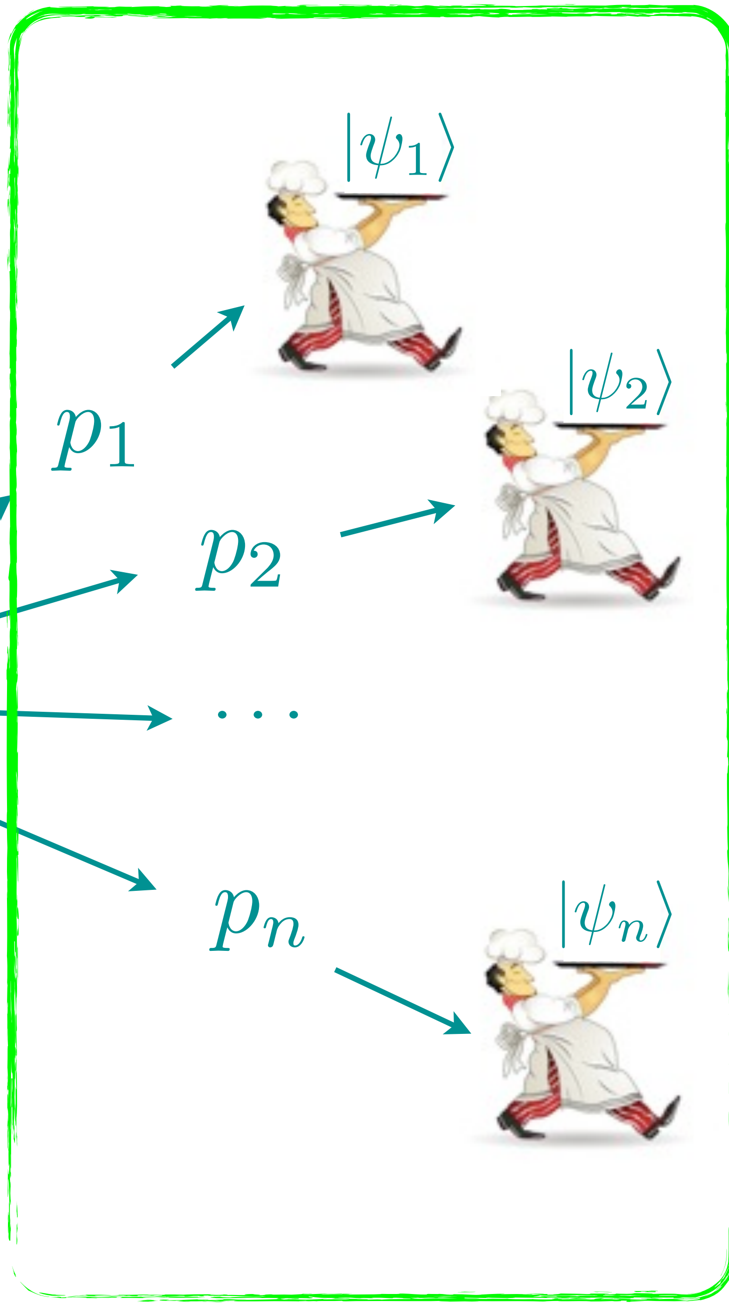
$|1\rangle = |\updownarrow\rangle$

$|0\rangle = |\leftrightarrow\rangle$

STATES

Statistical mixture (ensemble)

fuzzy instructions ...



p_j = probability that the system S is in $|\psi_j\rangle$

$$\{|\psi_j\rangle; p_j\}$$

DENSITY OPERATOR

$$\rho = \sum_{j=1}^{n} p_j |\psi_j\rangle\langle\psi_j|$$

$$\rho \geqslant 0$$

$$\mathrm{Tr}[\rho] = 1$$

STATES

SOME PROPERTIES:

The correspondence between ensembles and density matrices in general is NOT unique. Different ensembles may be associated to a same DM. Each of them provides a different UNRAVELLING of ρ.
A special unravelling is provided by the spectral decomposition of the ρ .

ENSEMBLE 1

$\{|\psi_j\rangle; p_j\}$

$\{|\psi'_j\rangle; p'_j\}$

ENSEMBLE 2

$\rho \longleftarrow \{|j\rangle; \lambda_j\}$

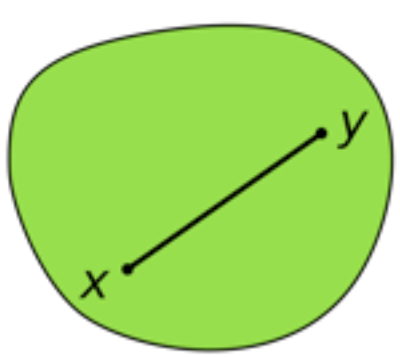
SPECTRAL DECOMPOSITION: it is formed by the eigenvectors and eigenvalues of ρ

(i) All density matrices satisfy the condition $\mathrm{Tr}[\rho^2] \leqslant 1$ with IDENTITY if and only if the state is pure, i.e. $\rho = |\psi\rangle\langle\psi|$

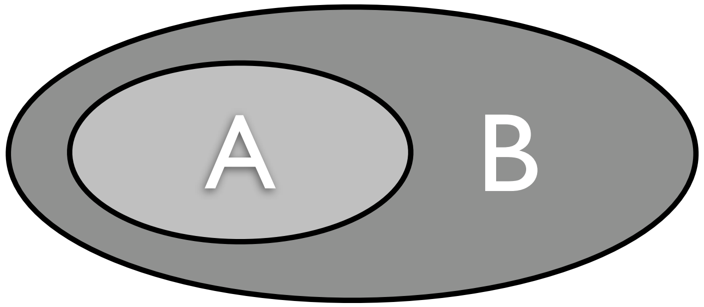
(ii) The set $\mathfrak{S}(\mathcal{H})$ of the density matrices is convex. Pure states are extremal elements of such set.

$$\rho_j \in \mathfrak{S}(\mathcal{H}) \quad \Longrightarrow \quad \sum_j p_j \rho_j \in \mathfrak{S}(\mathcal{H})$$

probability distribution



COMPOSITE SYSTEMS



$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$ tensor product space

Given ρ_{AB} a state of AB the state of A and B are expressed by the reduced density matrices (~same role of the marginal distributions in statistical mechanics).

$$\rho_A = \mathrm{Tr}_B[\rho_{AB}]$$ partial trace over B

$$\rho_B = \mathrm{Tr}_A[\rho_{AB}]$$ partial trace over A

The state ρ_{AB} is factorizable if it can be expressed as a tensor product of states of the subsystems

$$\rho_{AB} = \rho_A \otimes \rho_B$$

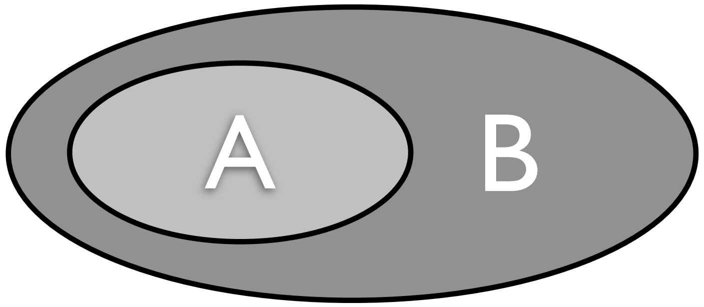
The state ρ_{AB} is said to be separable if it can be expressed as a convex combination of factorizable states

$$\rho_{AB} = \sum_j p_j \rho_A(j) \otimes \rho_B(j)$$

A state ρ_{AB} which is NOT separable is said to be entangled

$$(|00\rangle + |11\rangle)/\sqrt{2}$$

PURIFICATION (~inverse of reduction)



Given ρ_A state of A, a PURIFICATION of it, is any PURE state of AB $|\Psi_\rho\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$\rho_A = \mathrm{Tr}_B[|\Psi_\rho\rangle_{AB}\langle\Psi_\rho|]$$

e.g. given the spectral decomposition of ρ_A

$$\rho_A = \sum_j \lambda_j \; |j\rangle_A\langle j| \qquad \Longrightarrow \qquad \boxed{|\Psi_\rho\rangle_{AB} = \sum_j \sqrt{\lambda_j}|j\rangle_A \otimes |\phi_j\rangle_B}$$

$${}_B\langle\phi_j|\phi_{j'}\rangle_B = \delta_{jj'}$$

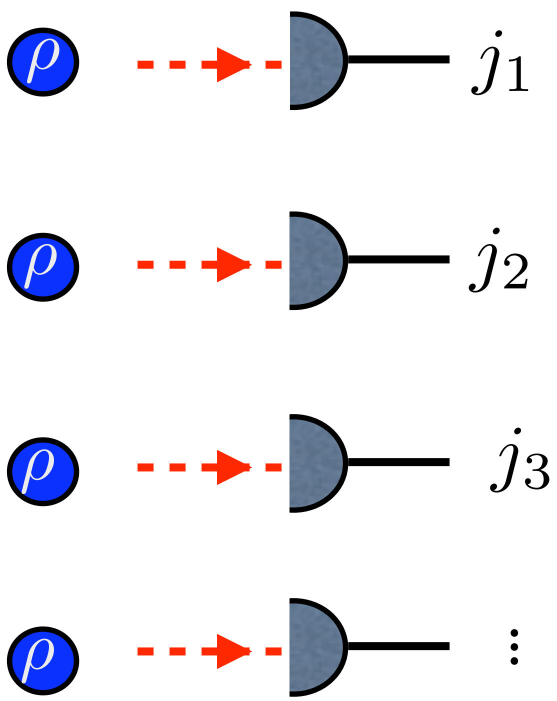
All PURIFICATIONS are connected via ISOMETRY acting on the ANCILLARY system

$$|\Psi'_\rho\rangle_{AB} = (I_A \otimes U_B)|\Psi_\rho\rangle_{AB}$$


$|\psi\rangle$


Measurements in QM are probabilistic processes:

we can only assign the (conditional) probabilities that a certain outcome will occur when performing a *given measurement* on a *given state*.

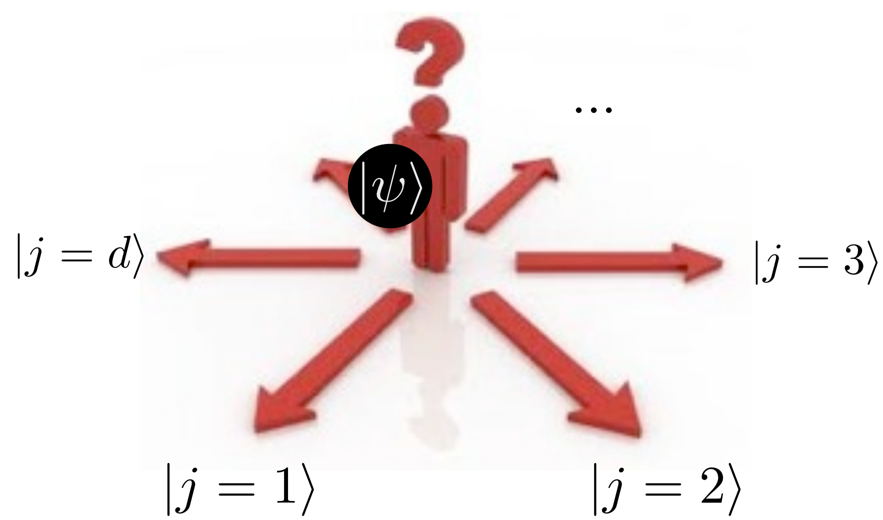


$p(j|\rho)$ (conditional) probability of getting the outcome j when performing the measurement on the state ρ

PROJECTIVE (or von Neumann) MEASUREMENTS: (simplest and more fundamental form of quantum measurements)

A Projective Measurement (PM) *tries* to identify the state $|\psi\rangle$ of the system among a collection of orthonormal configurations (basis of S):

$$\{|j\rangle\}_{j=1,\cdots,d} \qquad\qquad \langle j|j'\rangle = \delta_{jj'} \ .$$



NB: $|\psi\rangle$ needs not be one of the elements of the basis.

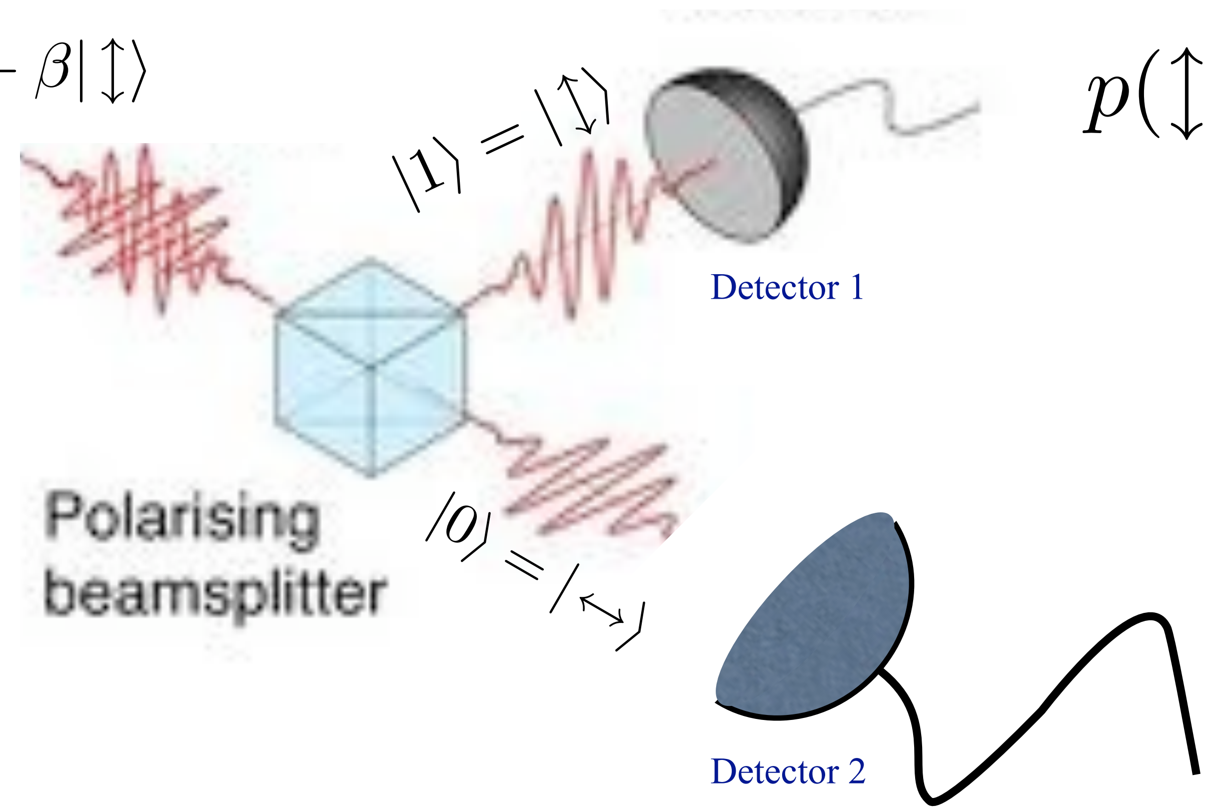
BORN RULE

$$p(j|\psi) = |\langle j|\psi\rangle|^2 \ ,$$

(for mixed state $p(j|\rho) = \langle j|\rho|j\rangle$).

Example: *polarized single photon*

$$|\psi\rangle = \alpha|\leftrightarrow\rangle + \beta|\updownarrow\rangle$$



$$p(\updownarrow|\psi) = |\beta|^2$$

$$p(\leftrightarrow|\psi) = |\alpha|^2$$

PROJECTIVE MEASUREMENTS can be described by assigning a collection of (possibly rank one) orthogonal *PROJECTORS* which partition the Hilbert space into mutually exclusive sectors,

$$\Pi_j = |j\rangle\langle j| \longrightarrow \begin{cases} \Pi_j \Pi_{j'} = \delta_{jj'}\, \Pi_j \\ \\ \sum_{j=1}^{d} \Pi_j = I \end{cases} \Rightarrow \boxed{p(j|\rho) = \langle j|\rho|j\rangle = \mathrm{Tr}[\Pi_j \rho]}$$

PROJECTIVE MEASUREMENTS as OBSERVABLES

(orthonormal) eigenstates

$$\Theta = \Theta^\dagger \quad \in \mathcal{L}(\mathcal{H}) \qquad \longrightarrow \qquad \Theta|j\rangle = \theta_j |j\rangle$$

(real) eigenvalues

$$\langle \Theta \rangle = \mathrm{Tr}[\Theta \rho] = \sum_j p(j|\rho)\ \theta_j$$ expectation value of the observable

$$\Delta\Theta = \sqrt{\sum_j p(j|\rho)\, [\theta_j - \langle\Theta\rangle]^2} = \sqrt{\langle \Theta^2 \rangle - \langle \Theta \rangle^2}$$ RMSE (standard deviation)

UNCERTAINTY RELATIONS

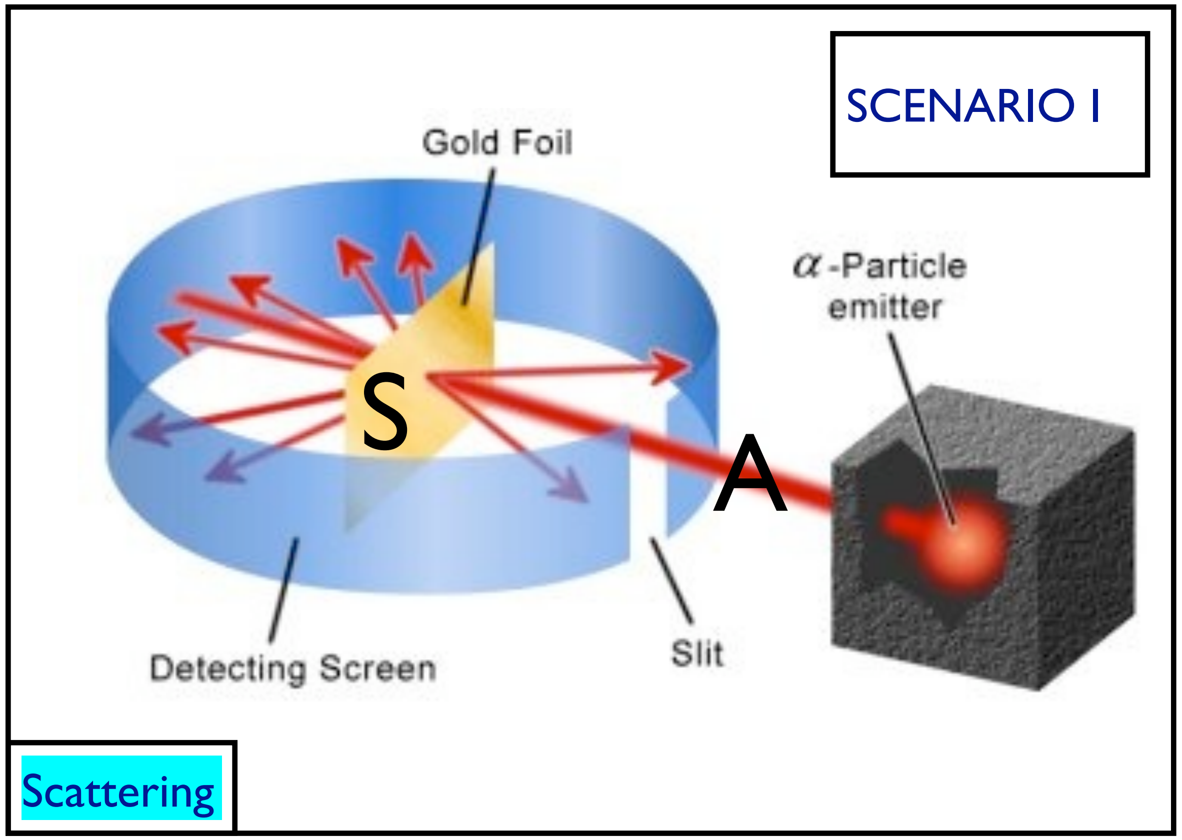
$$\Omega\Theta \neq \Theta\Omega$$

incompatible observables

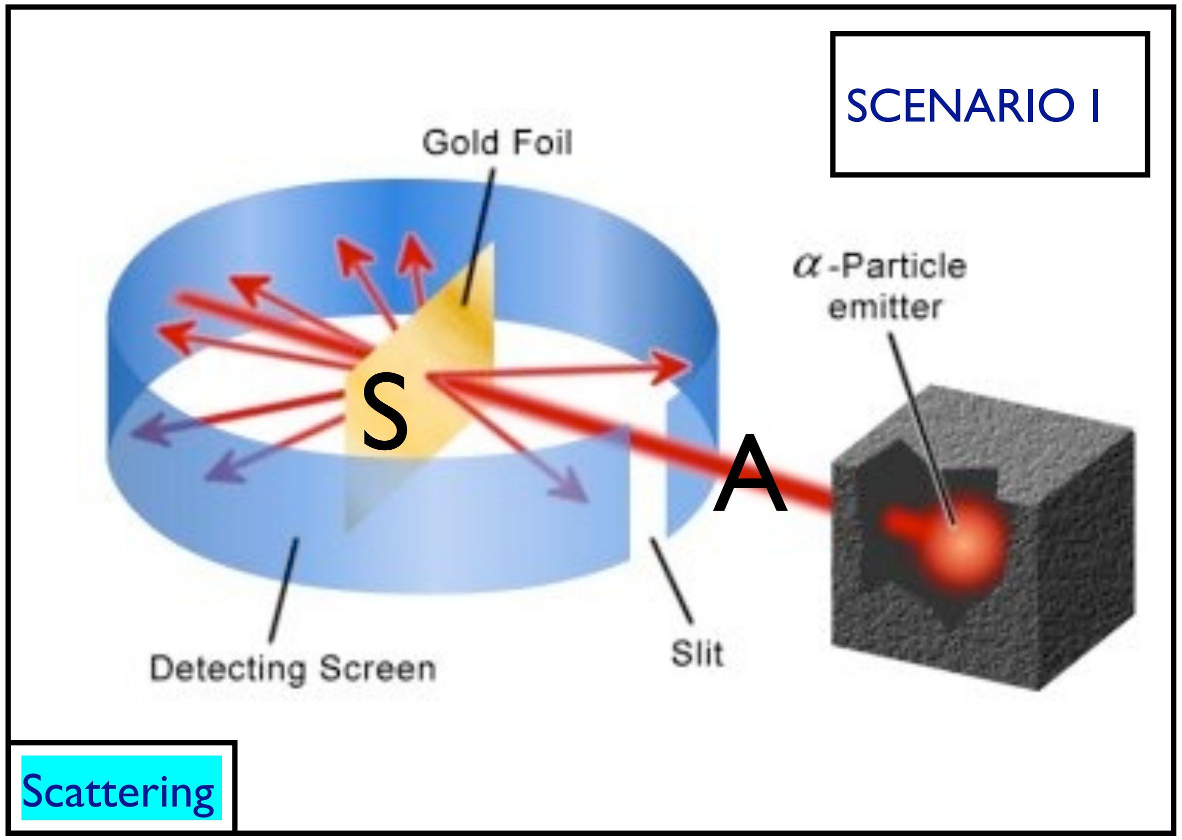
$$\Delta\Omega\Delta\Theta \geq \frac{|\langle[\Omega,\Theta]\rangle|}{2}$$

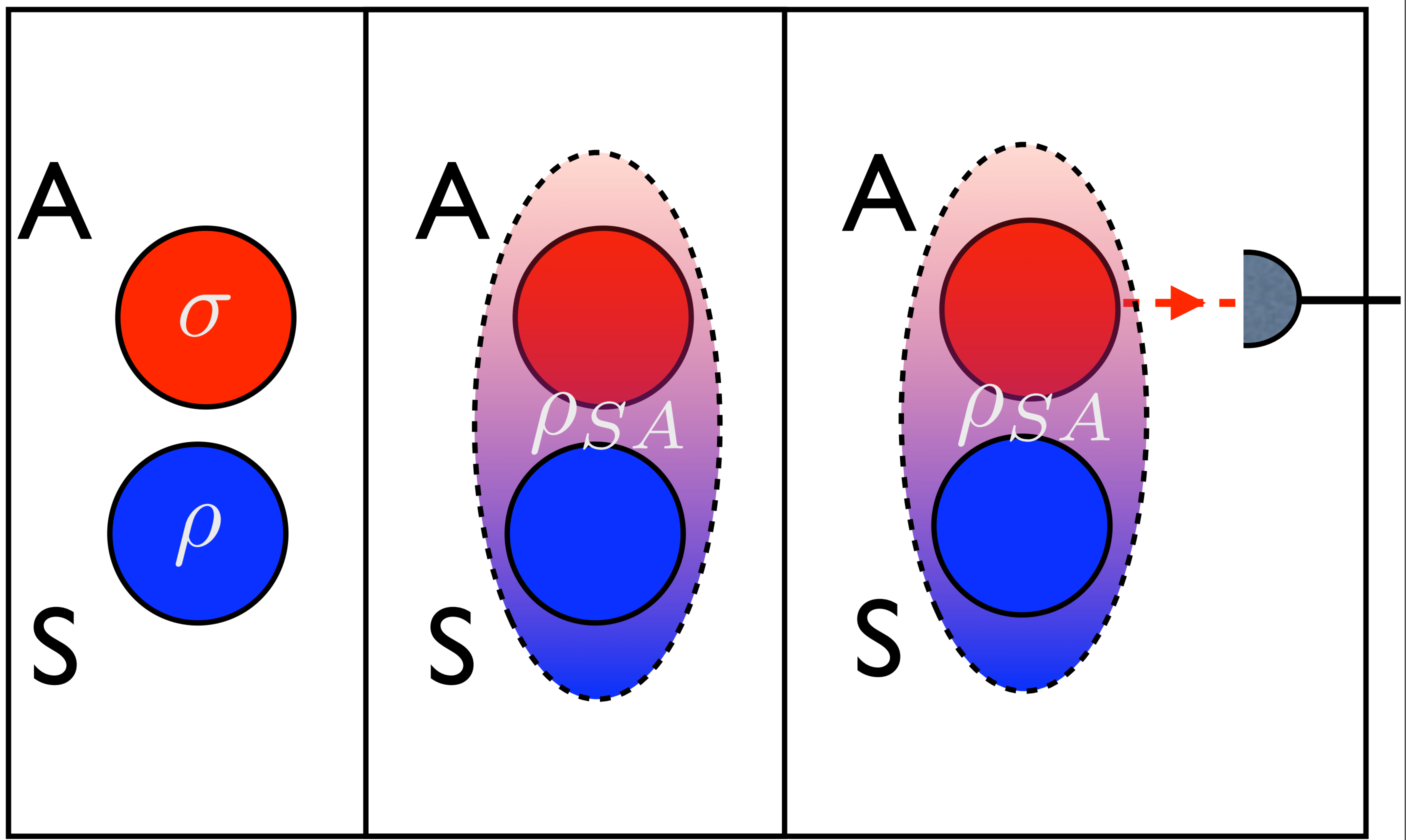
Robertson inequality

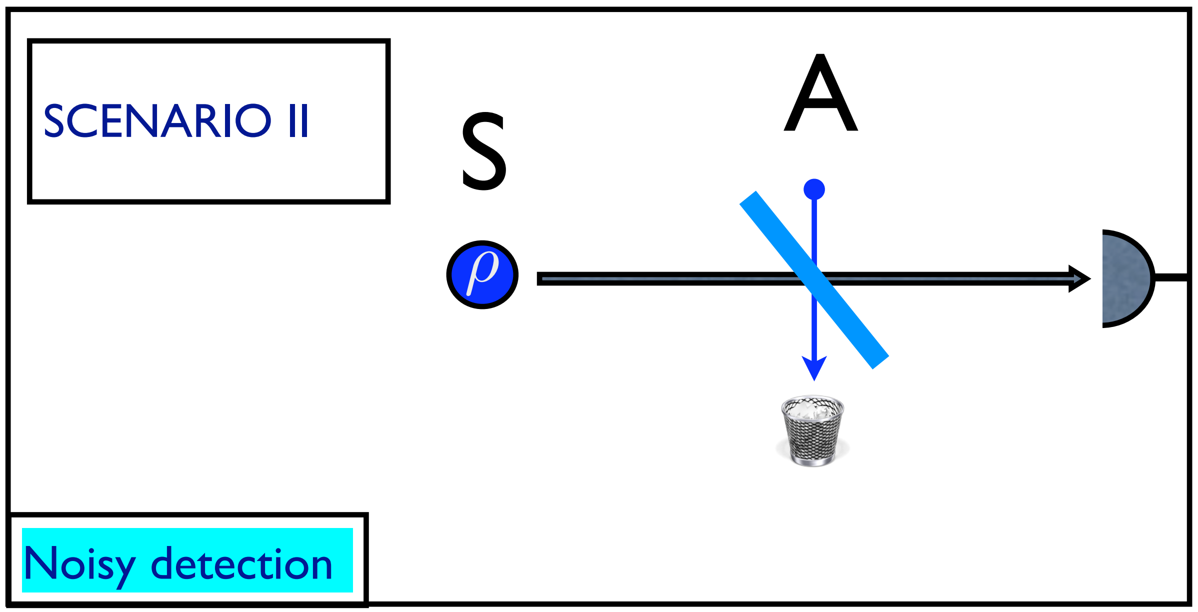
Beyond PROJECTIVE MEASUREMENTS: Indirect or mediated measurements

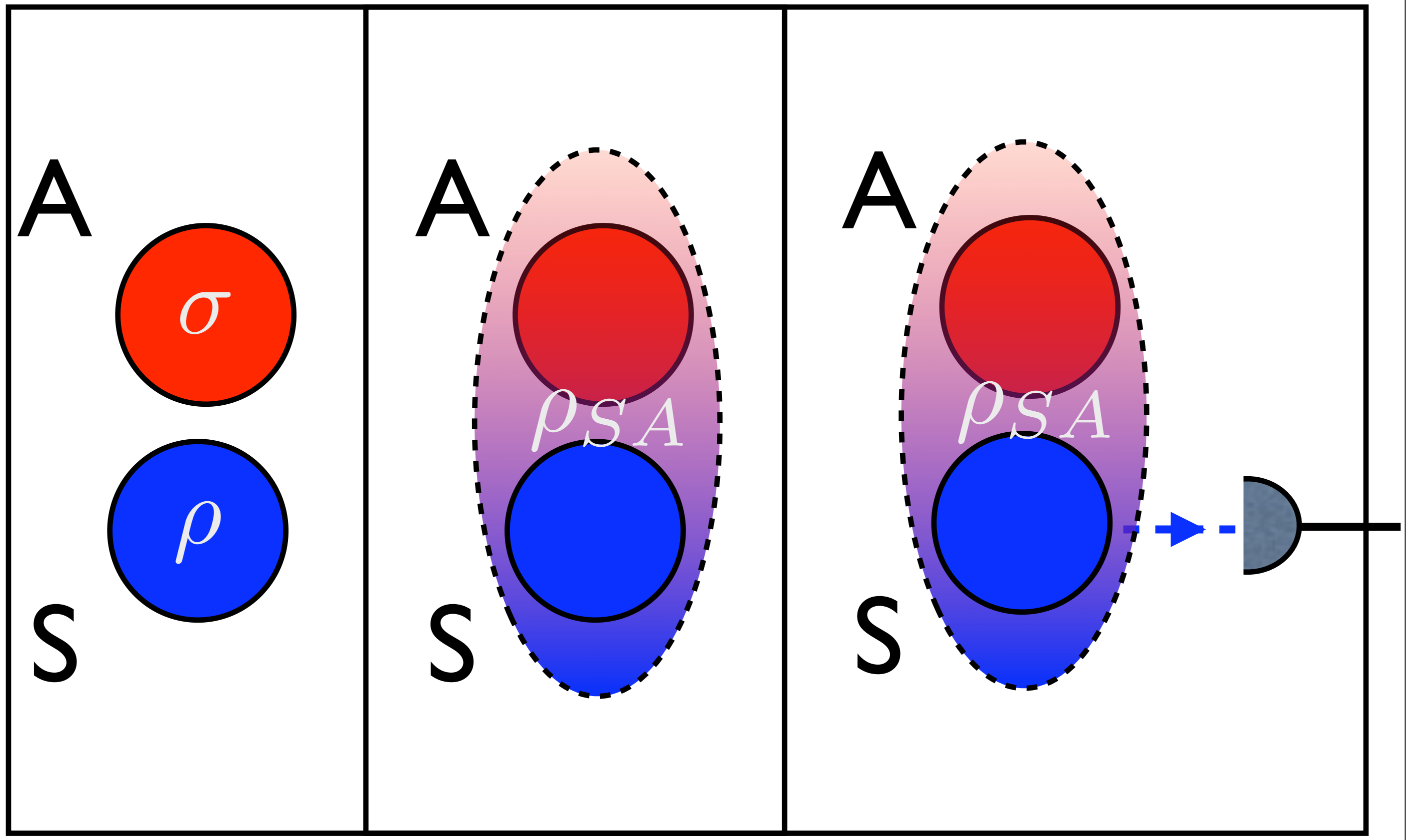


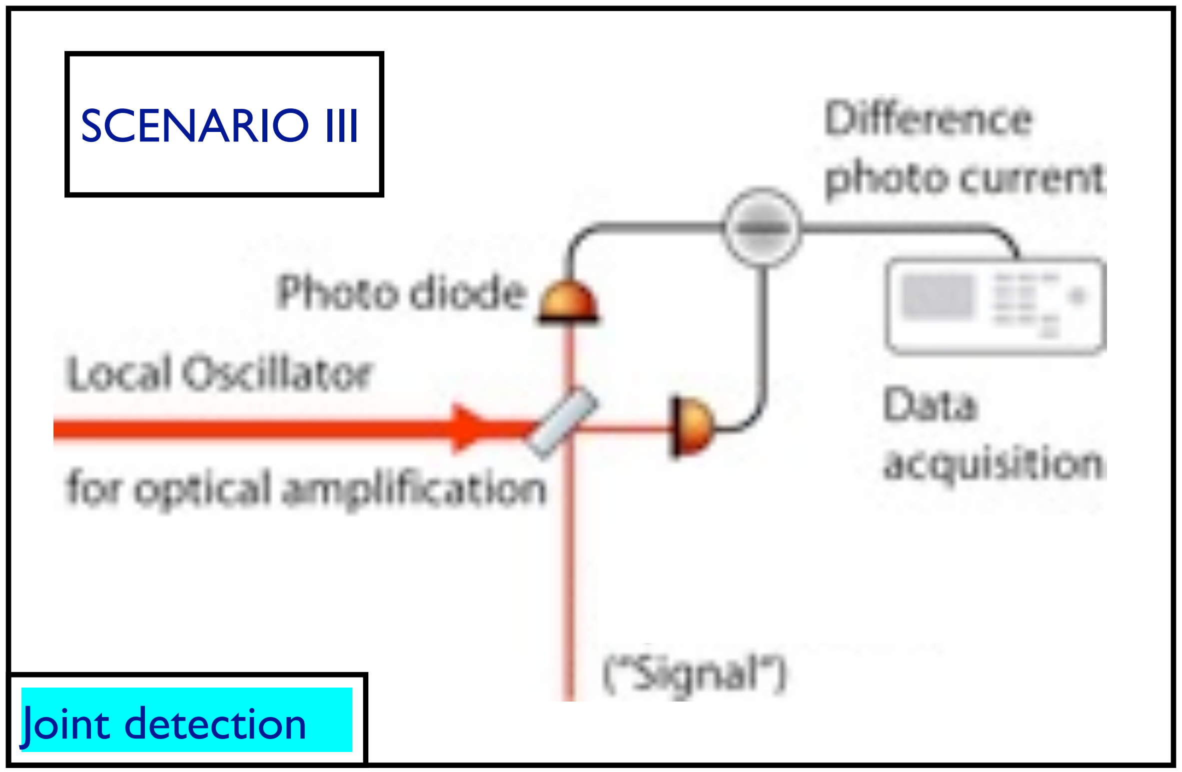
Beyond PROJECTIVE MEASUREMENTS: Indirect or mediated measurements

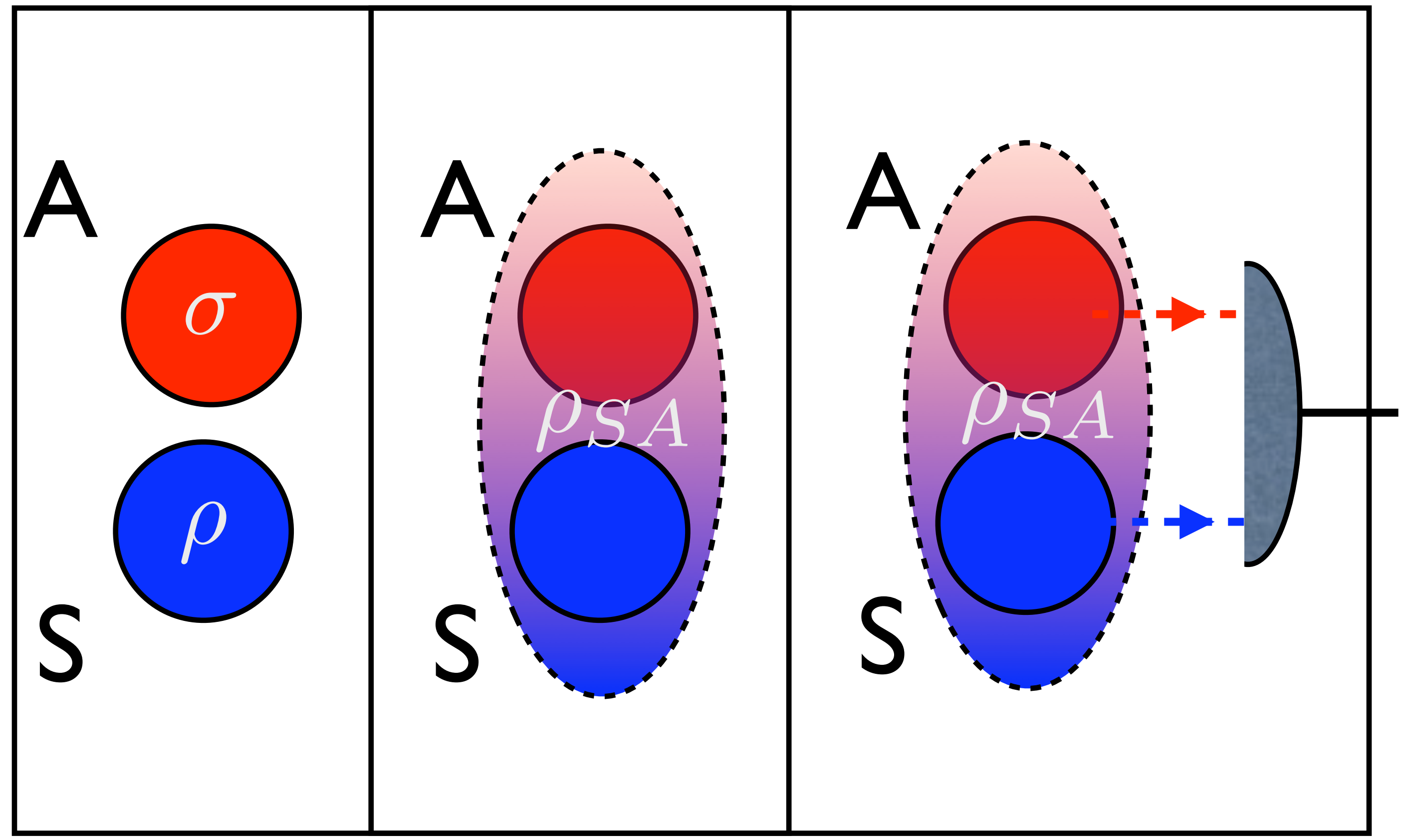




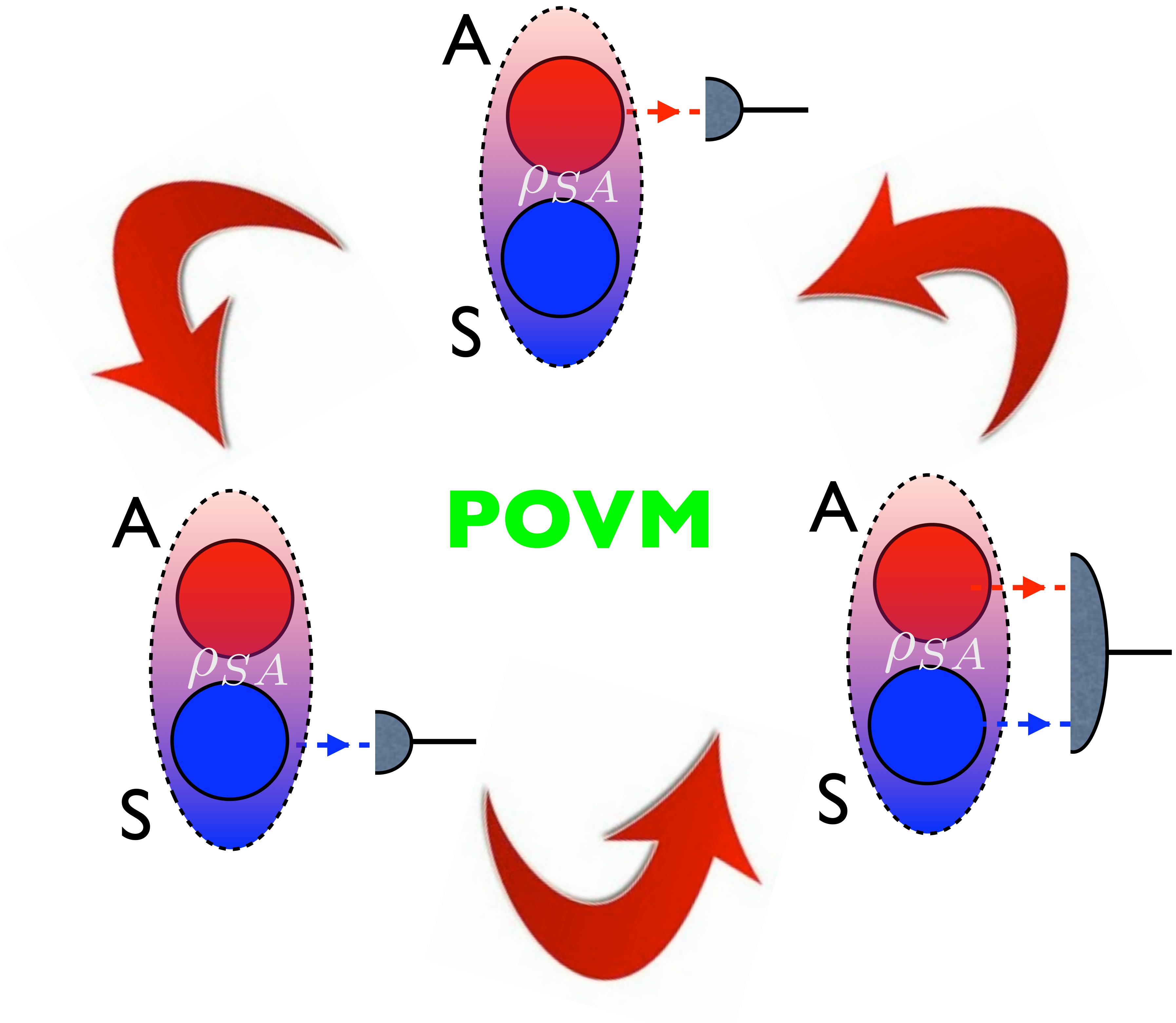

SCENARIO II
S
A
ρ
Noisy detection


A
S
σ
ρ
A
S
ρ_{SA}
A
S
ρ_{SA}


SCENARIO III
Difference photo current
Photo diode
Local Oscillator
for optical amplification
Data acquisition
("Signal")
Joint detection


A
S
σ
ρ
A
S
ρ_{SA}
A
S
ρ_{SA}

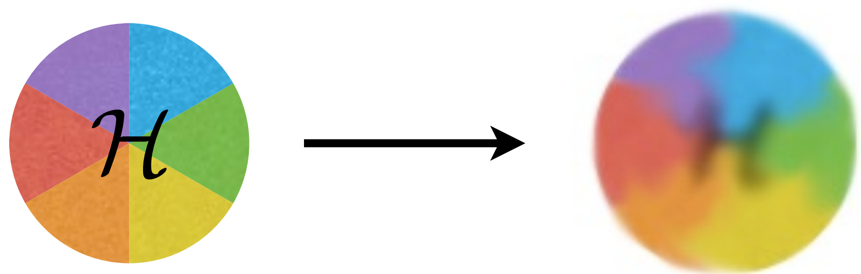
Beyond PROJECTIVE MEASUREMENTS: Indirect or mediated measurements



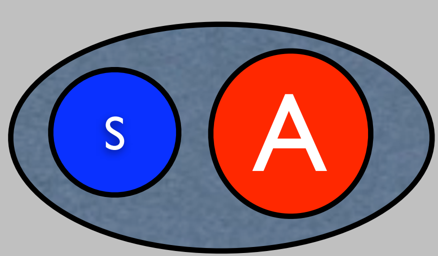
POVM (Positive Operator Valued Measurement): it is the most general form of quantum measurement. It is described by assigning a (normalized) collection of POSITIVE OPERATORS,

$$p(j|\rho) = \mathrm{Tr}[E_j \rho]$$

$$E_j \geqslant 0 \qquad \sum_{j=1}^{n} E_j = I$$

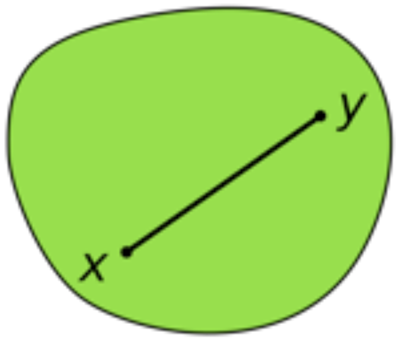


Via *Naimark extension* ALL POVMs can be described as a PM on a larger space



How many different TYPES of measurements do exist for a given quantum system S?

INFINITELY MANY in principle [e.g. already projective measurements are uncountable]. They form a convex manifold:

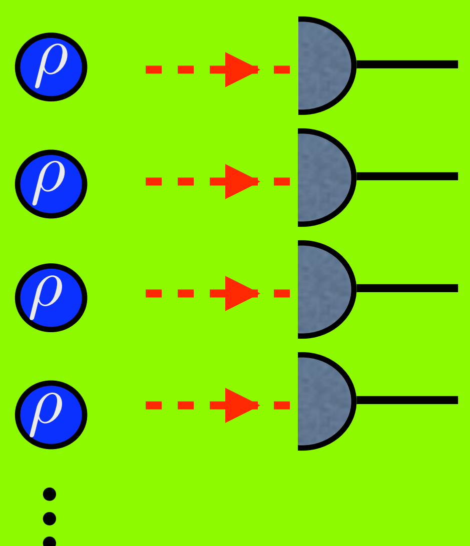


Of course ONLY FEW which are useful can be implemented in practice [which one depends upon the level of control you have on the system]!

Applications

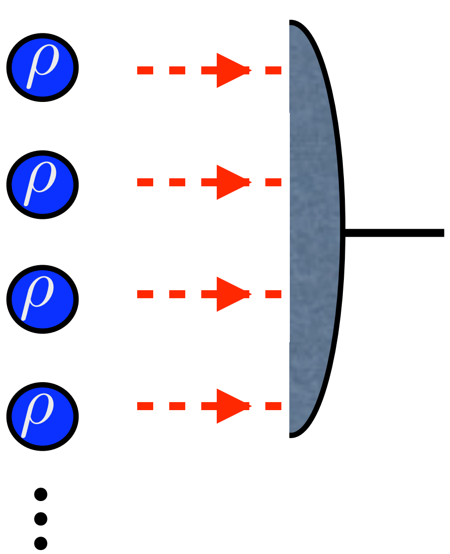
Application 0: Measuring Expectation Value of an OBSERVABLE

$$\langle \Theta \rangle = \mathrm{Tr}[\Theta\rho] = \sum_j p(j|\rho)\,\theta_j$$



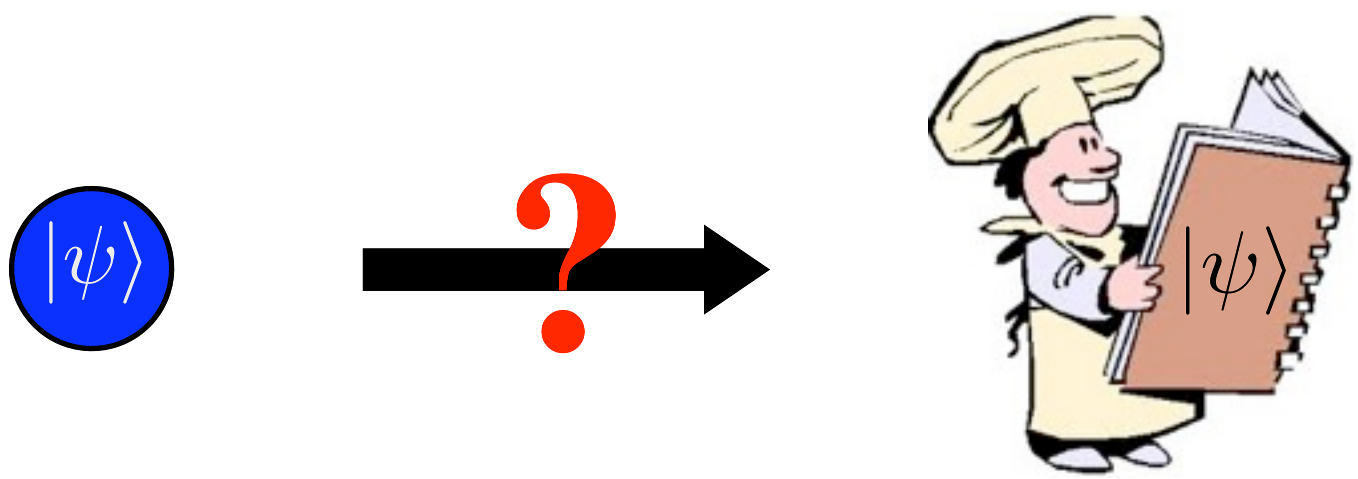
$$\left.\begin{array}{l}\theta_{j_1}\\ \theta_{j_2}\\ \theta_{j_3}\\ \vdots\end{array}\right\} \quad \Theta_N^{(est)} = \frac{1}{N}\sum_{n=1}^{N}\theta_{j_n} \quad \Longrightarrow \quad \langle \Theta \rangle \quad \text{(error goes down as } \simeq 1/\sqrt{N}\text{)}$$

data processing
(averaging)



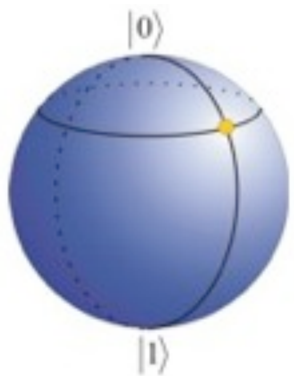
Can we improve? $\Rightarrow$ NO

Application 1: State Tomography
(reconstruction of a state)



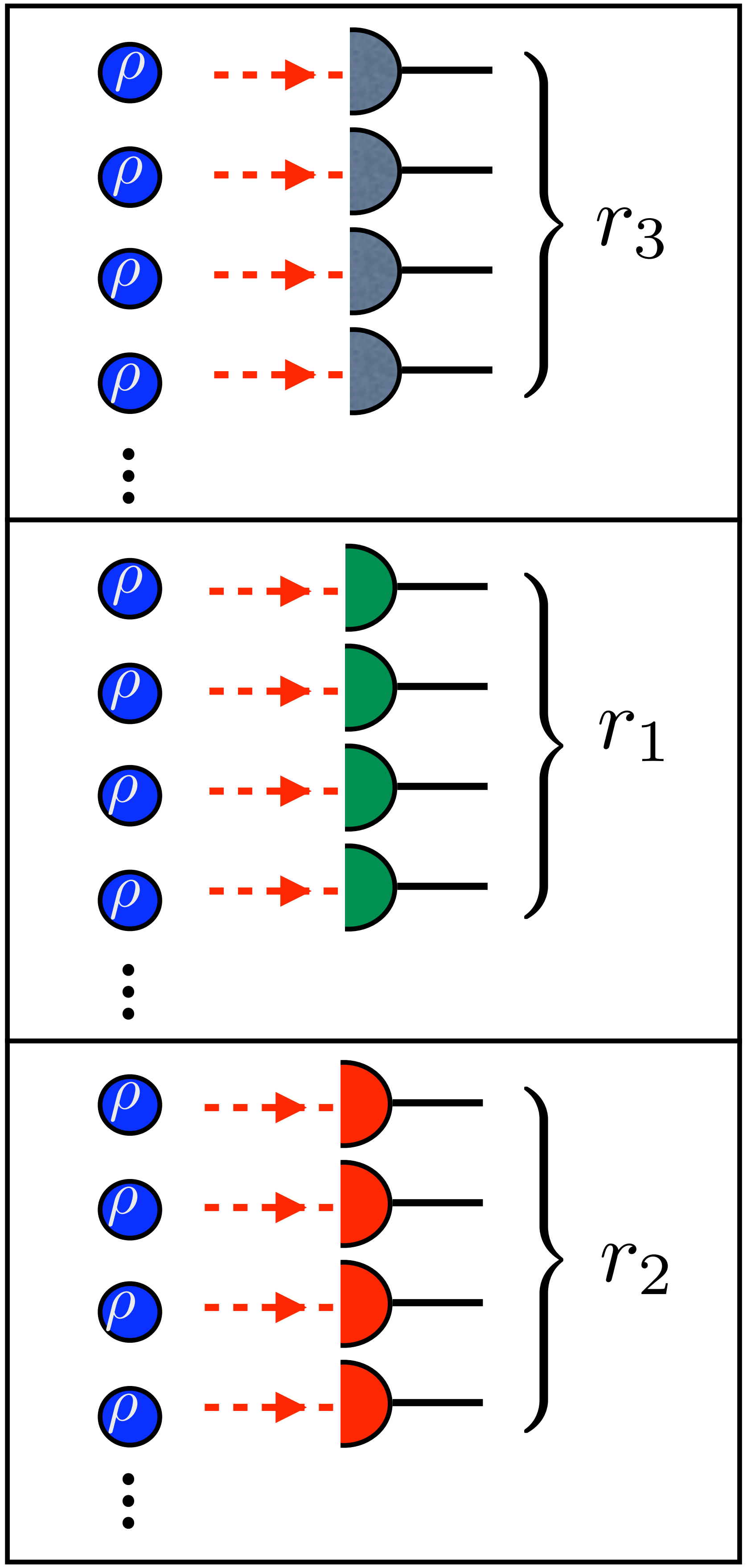
IMPOSSIBLE if WE HAVE A SINGLE COPY OF THE STATE [NO CLONING Wootters, Zurek Nature 299 (1982)]

Example I: *qubit tomography*



$$\rho = \rho(\vec{r}) = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$$

$\vec{r} \in \Re^3$ s.t. $|\vec{r}| \leqslant 1$



$$\{|0\rangle, |1\rangle\}$$

$$\{|+\rangle, |-\rangle\}$$

$|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$

$$\{|+i\rangle, |-i\rangle\}$$

$|\pm i\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$

Example II: *state tomography of a single optical mode [i.e. harmonic oscillator]*

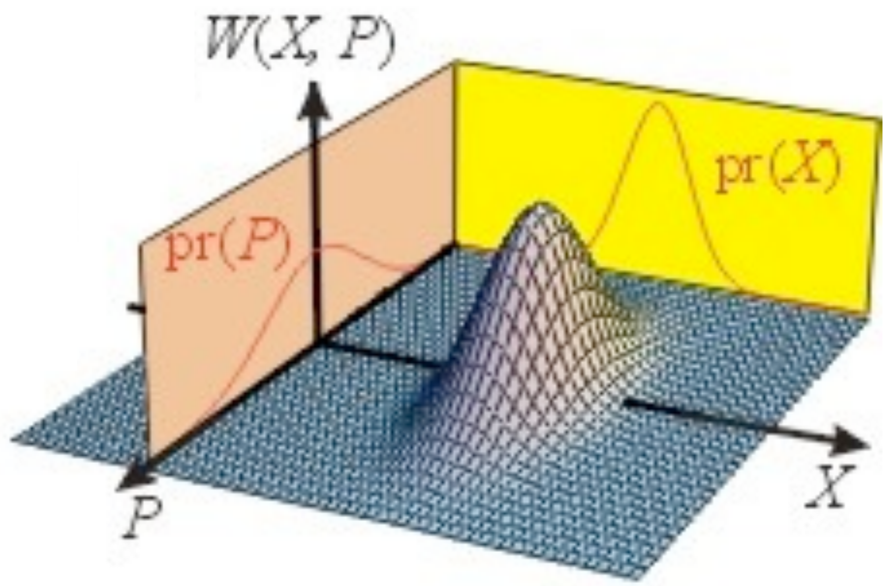
$$[a, a^\dagger] = 1 \qquad H = \hbar\omega[a^\dagger a + 1/2]$$

Wigner (quasi) distribution

$$X = \frac{a + a^\dagger}{\sqrt{2}}$$

$$P = i\frac{a - a^\dagger}{\sqrt{2}}$$

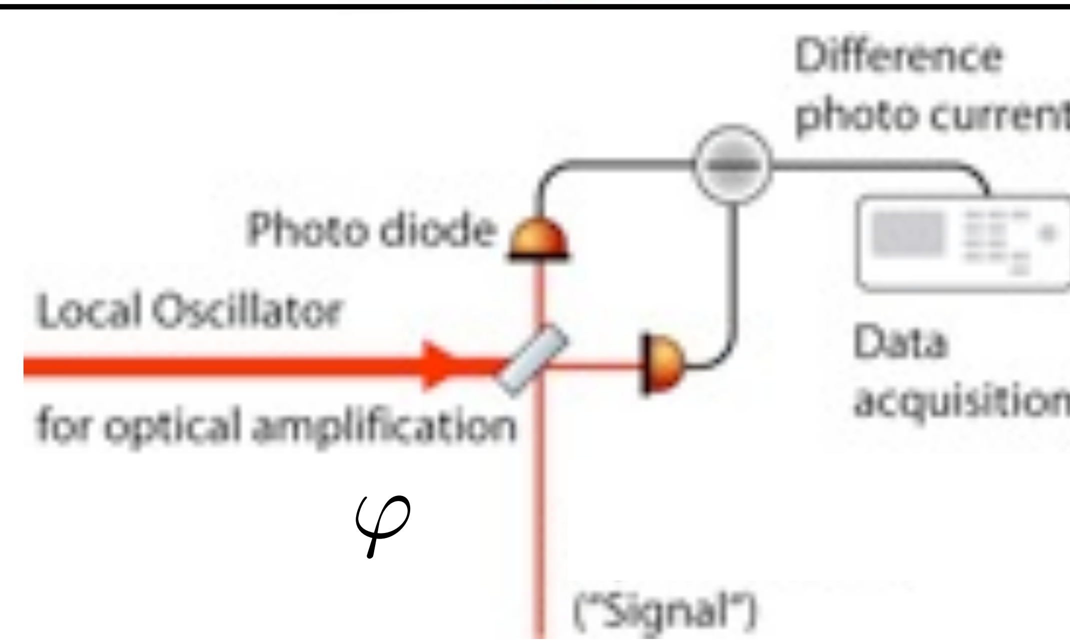
$$[X, P] = i$$



X-P phase space

$$X_\varphi = \frac{e^{i\varphi}a + e^{-i\varphi}a^\dagger}{\sqrt{2}}$$

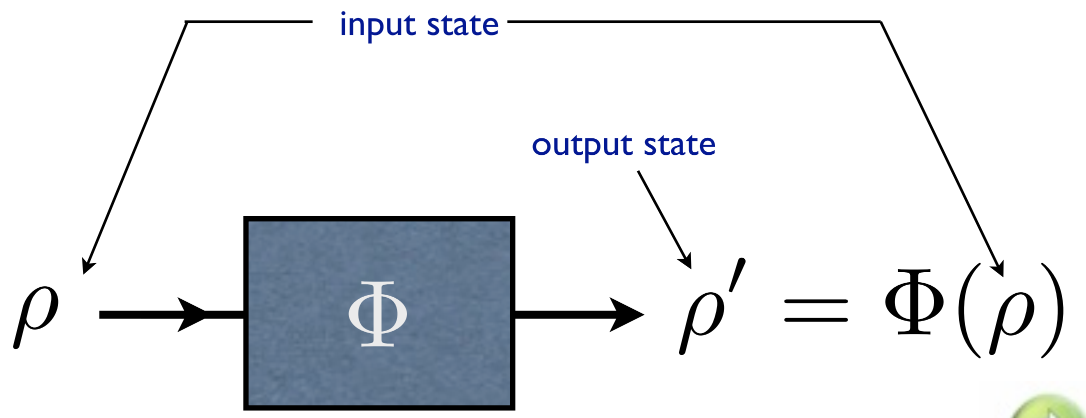
field "quadrature"



Homodyne detection

see D'Ariano, Paris, Sacchi quant-ph/0302028 for a review

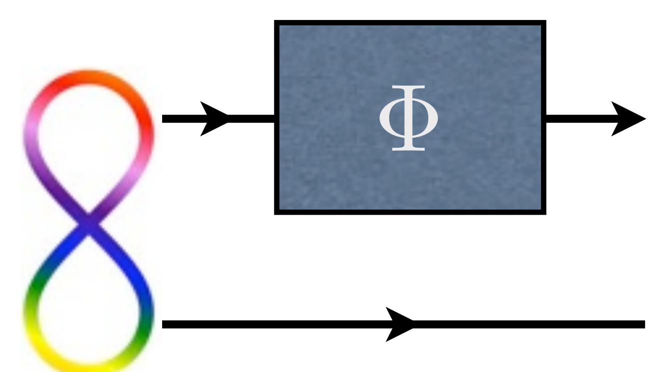
Application II: Process Tomography
(reconstruction of a transformation)



The most general discrete-time evolution of a quantum system is described by assigning a linear map (channel) Φ which connects the input state ρ of the system to its output counterpart.

Option 1: prepare a selected collection of input states and do state tomography on the associated output [due to the linearity of the problem, we need only a finite number of selected inputs].

Option 2 (via *Choi-Jamiolkowski isomorphism*): create an entangled input state and apply the map the to one of the two components. Then do state tomography on the output. [Here we use ONLY one input].



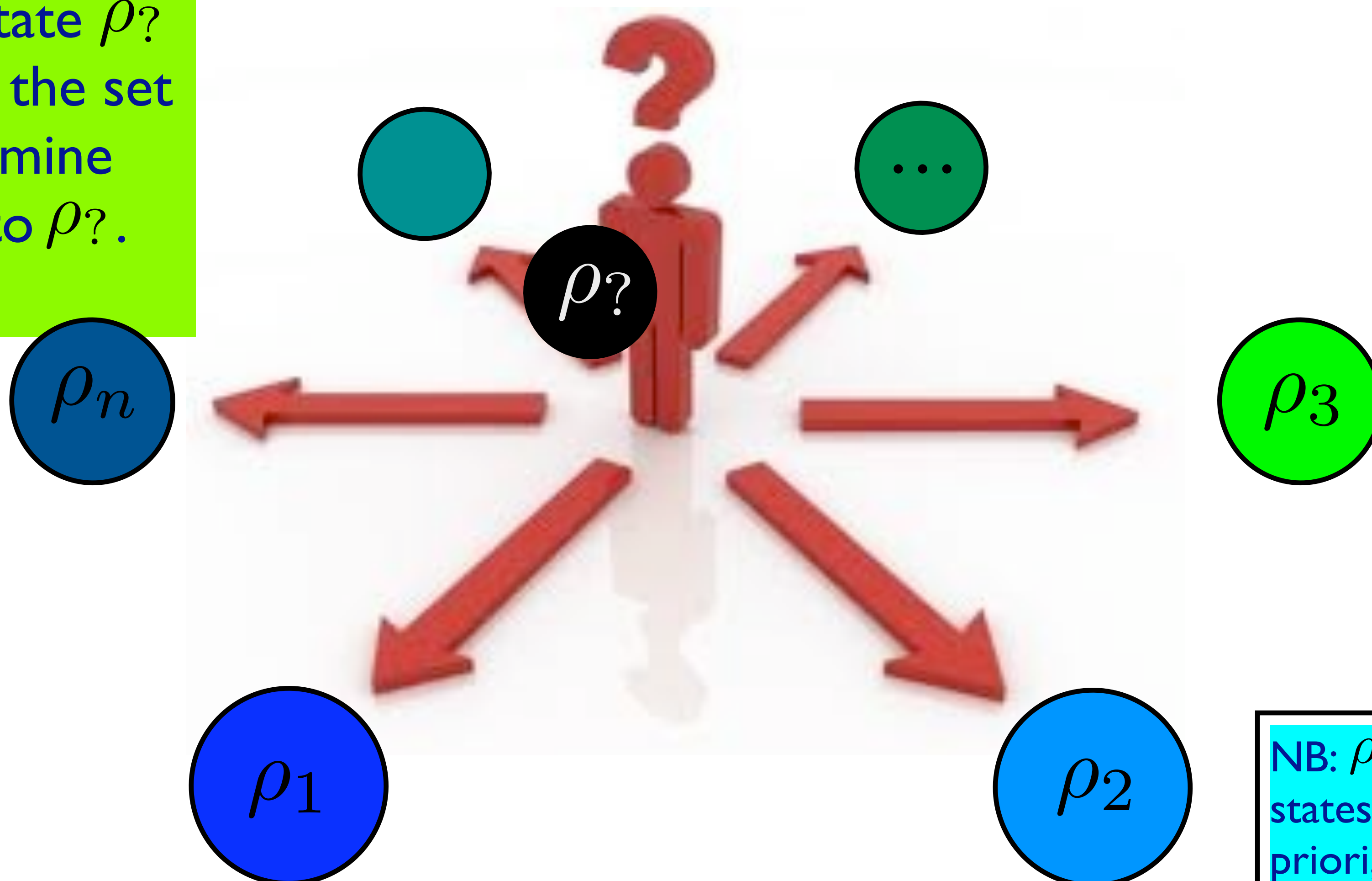
state tomography of the joint state

Application III: State Discrimination

Given a finite collection of possible states,

$$\rho_1, \rho_2, \cdots, \rho_n$$

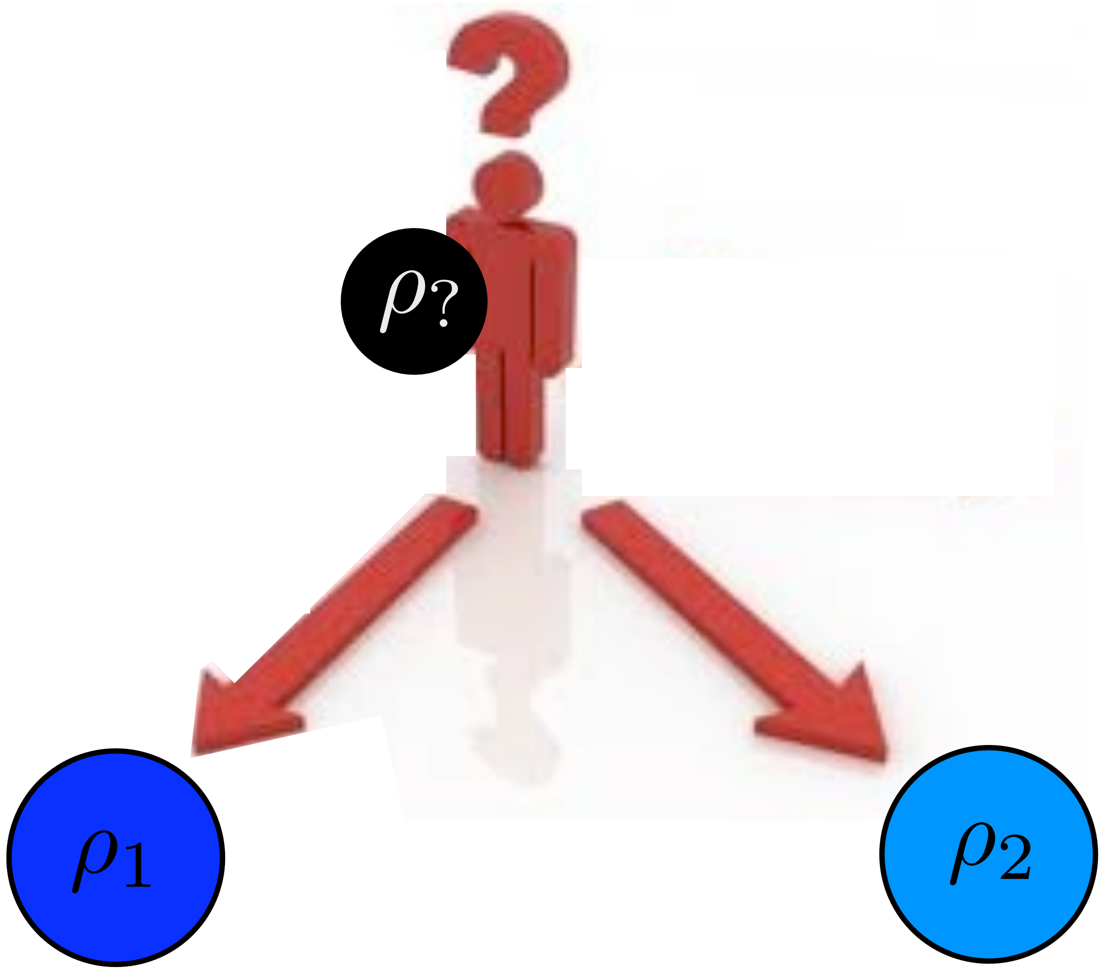
and a *single copy* of a state $\rho_?$ extracted randomly from the set of possible states, determine which one correspond to $\rho_?$.



NB: $\rho_?$ is one of the selected states $\rho_1, \rho_2, \cdots, \rho_n$ but, a priori, we don't know which one.

Find the (optimal) POVM which gives the best chance of success [e.g. the lowest error probability]

Example I: *two-state discrimination*



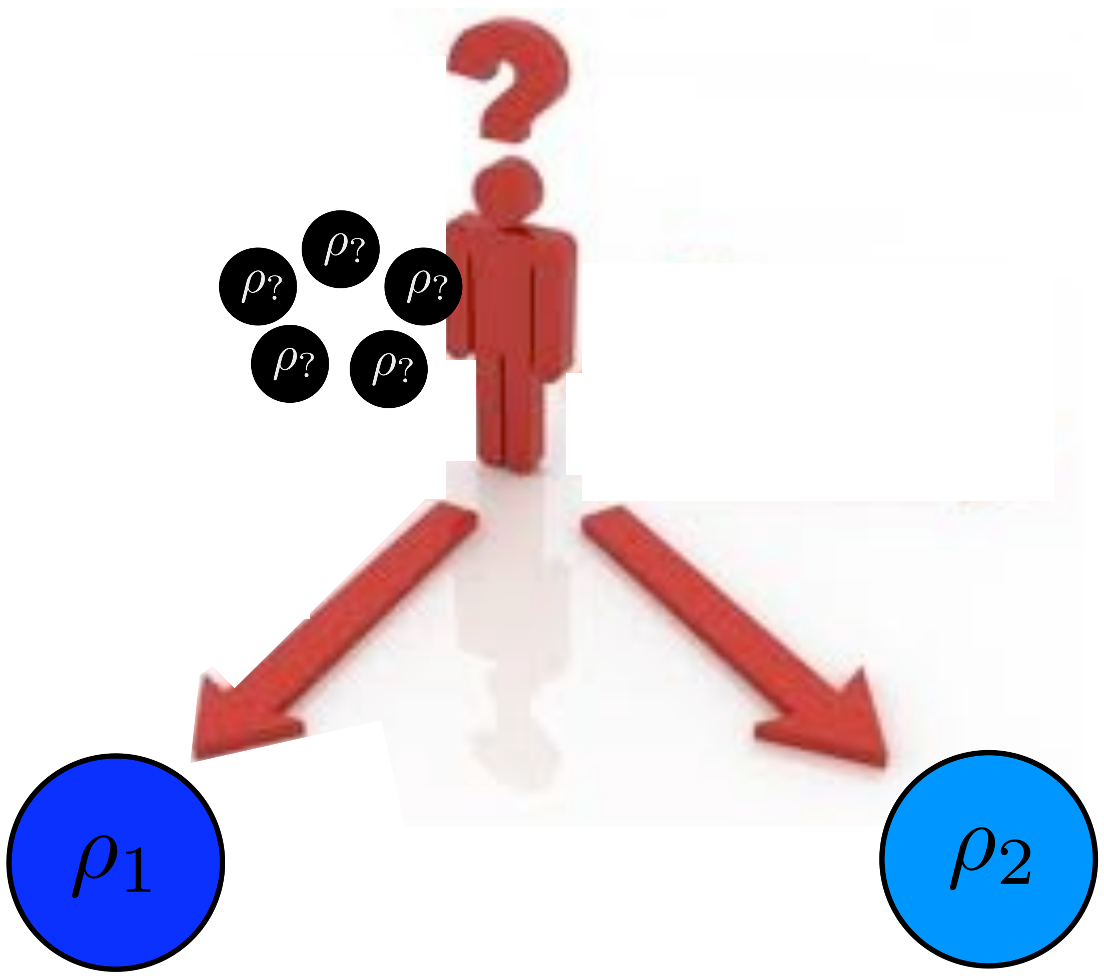
$$P_E = \frac{1 - \|\rho_1 - \rho_2\|_1/2}{2}$$

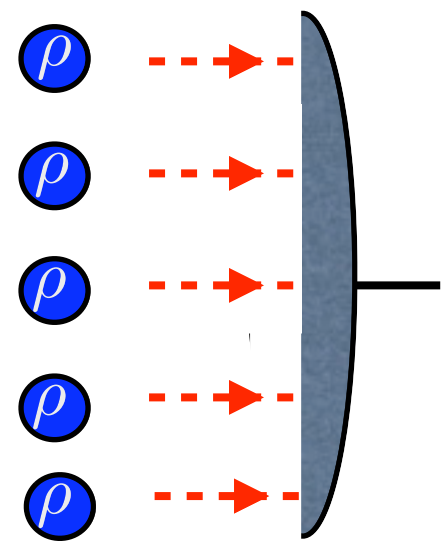
$$\|\Theta\|_1 = \mathrm{Tr}[\sqrt{\Theta^\dagger \Theta}]$$

TRACE DISTANCE $\quad D(\rho_1, \rho_2) = \frac{1}{2}\|\rho_1 - \rho_2\|_1$

(i) symmetric,
(ii) positive semi-definite and nullifies iff the two states coincides,
(iii) satisfies the triangular inequality.

Example I: *two-state discrimination with N identical copies of the unknown state*





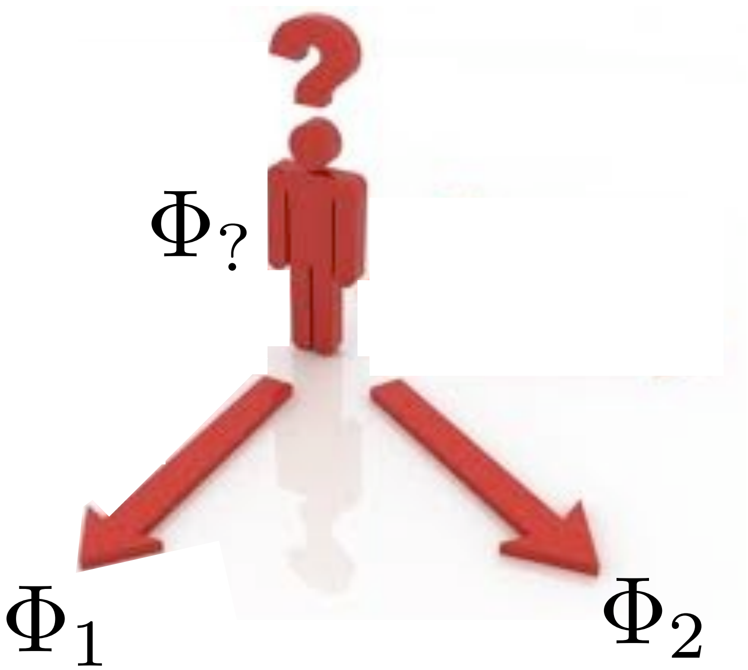
QUANTUM CHERNOFF BOUND

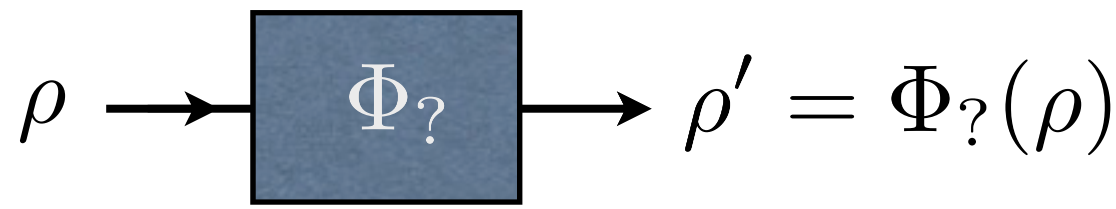
$$P_E^{(N)} = \frac{1 - \|\rho_1^{\otimes N} - \rho_2^{\otimes N}\|_1/2}{2} \lesssim \exp[-N\,\xi_{QCB}]$$

$$\xi_{QCB} = -\log\left\{\min_{0\le s\le 1} \mathrm{Tr}[\rho_1^s \rho_2^{(1-s)}\right\}$$

Audenaert et al. Phys. Rev. Lett. 98, 160501 (2007)

Application III: Process Discrimination

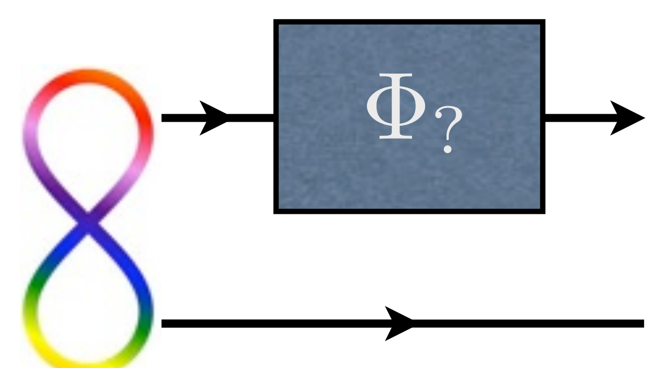




- Find optimal input (optimal POVM is known)

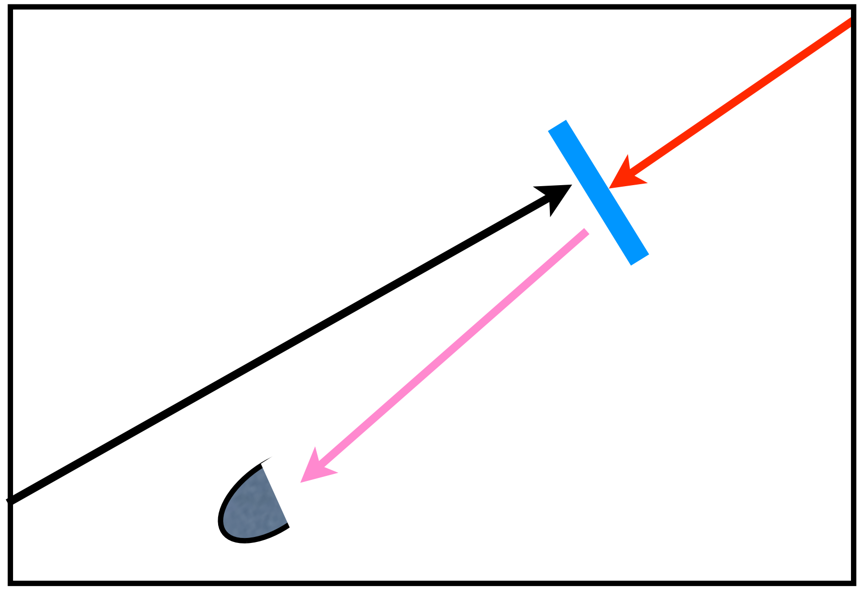
- Optimize the problem with realistic resource

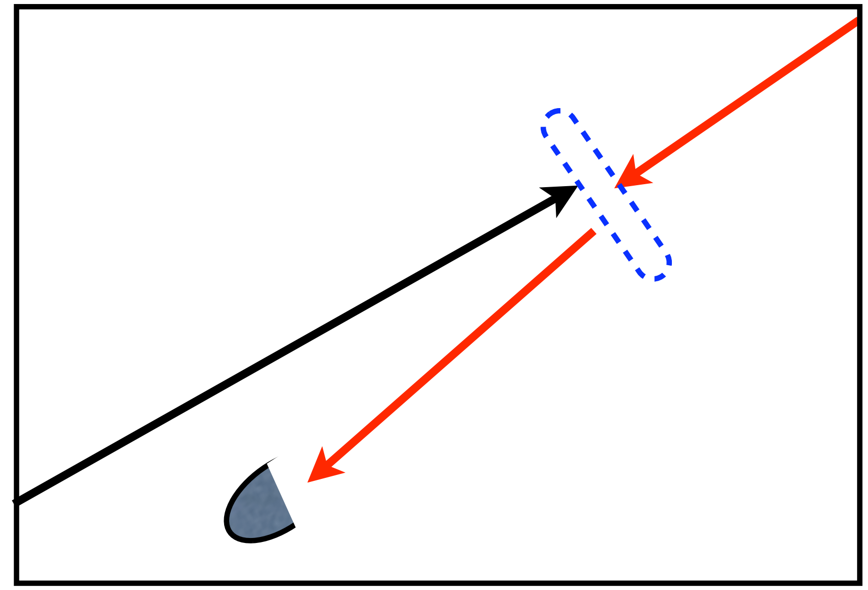
- Use side channels + entangled probes

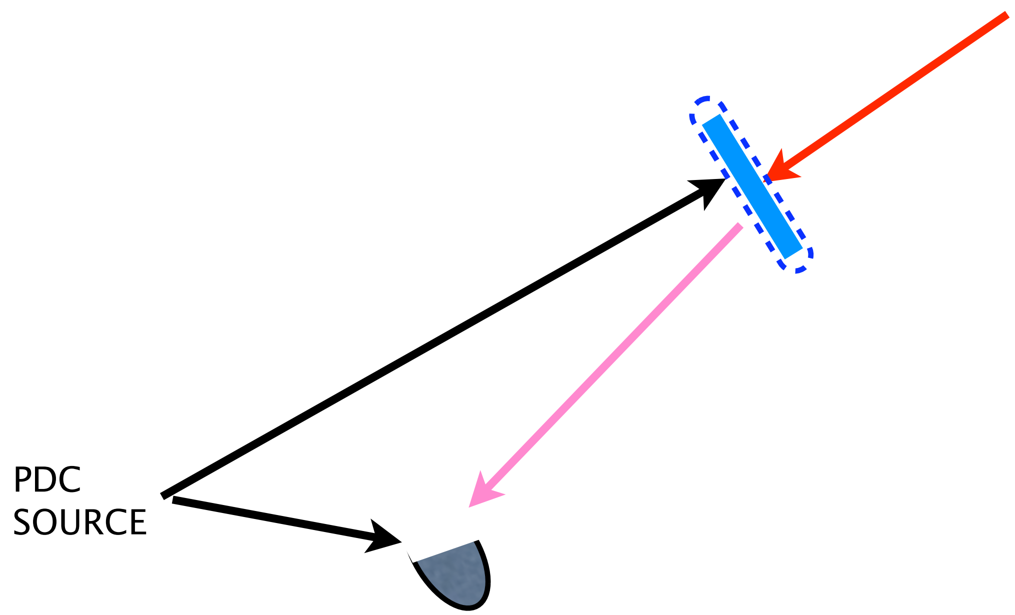


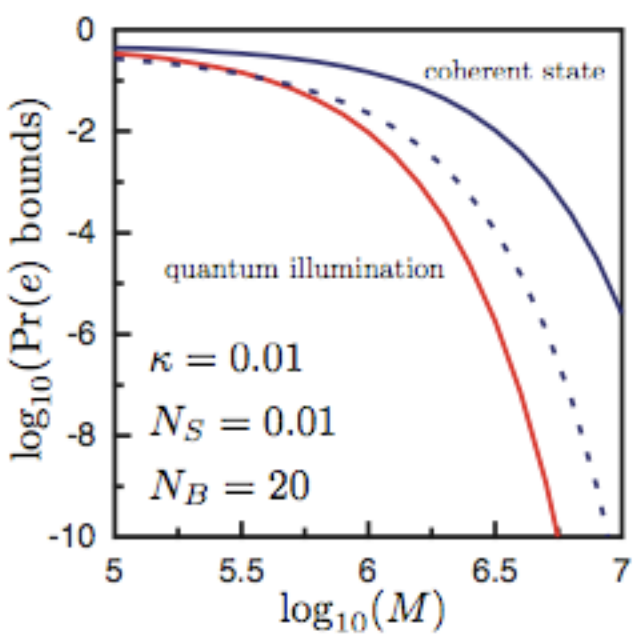
Quantum Illumination

Tan, et al. Phys. Rev. Lett. (2008)



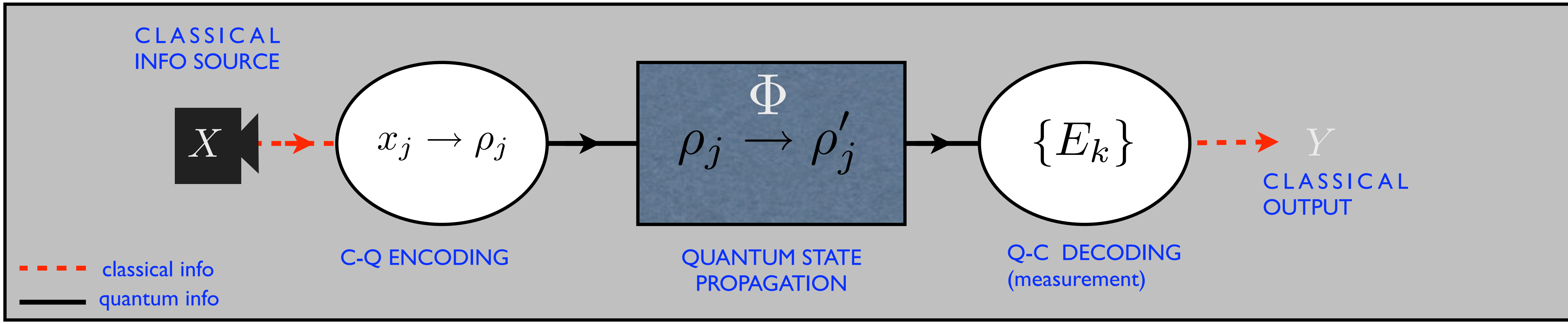






Application IV: Quantum Communication

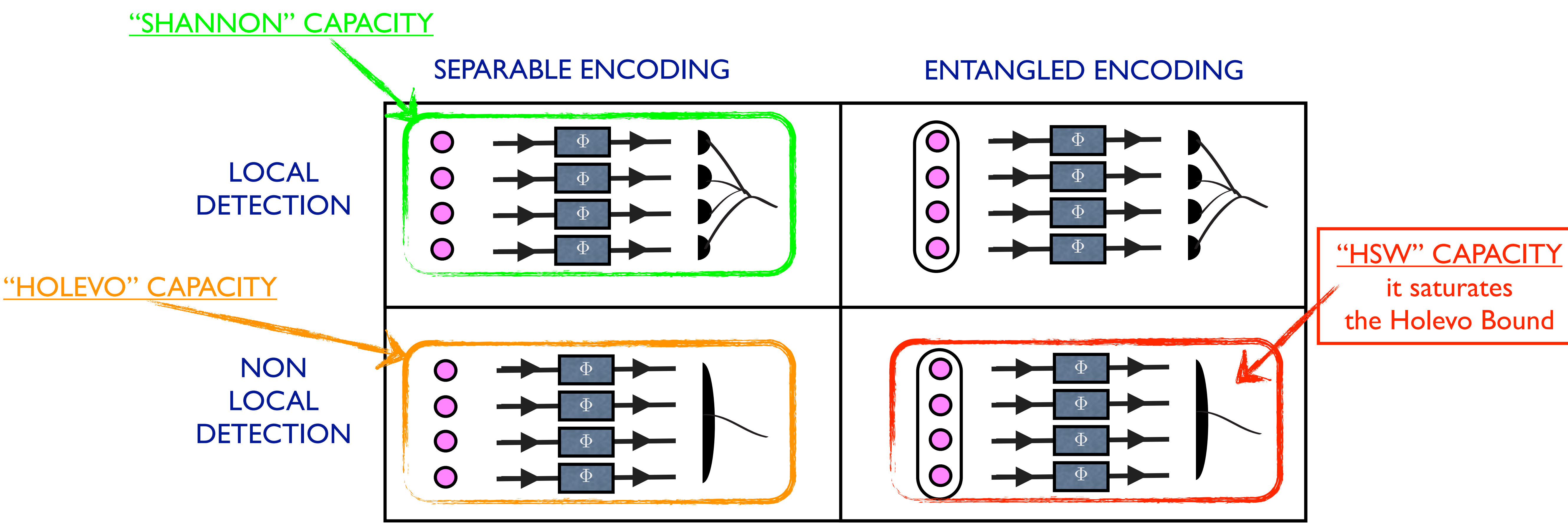
Transferring Classical Info over a quantum channel



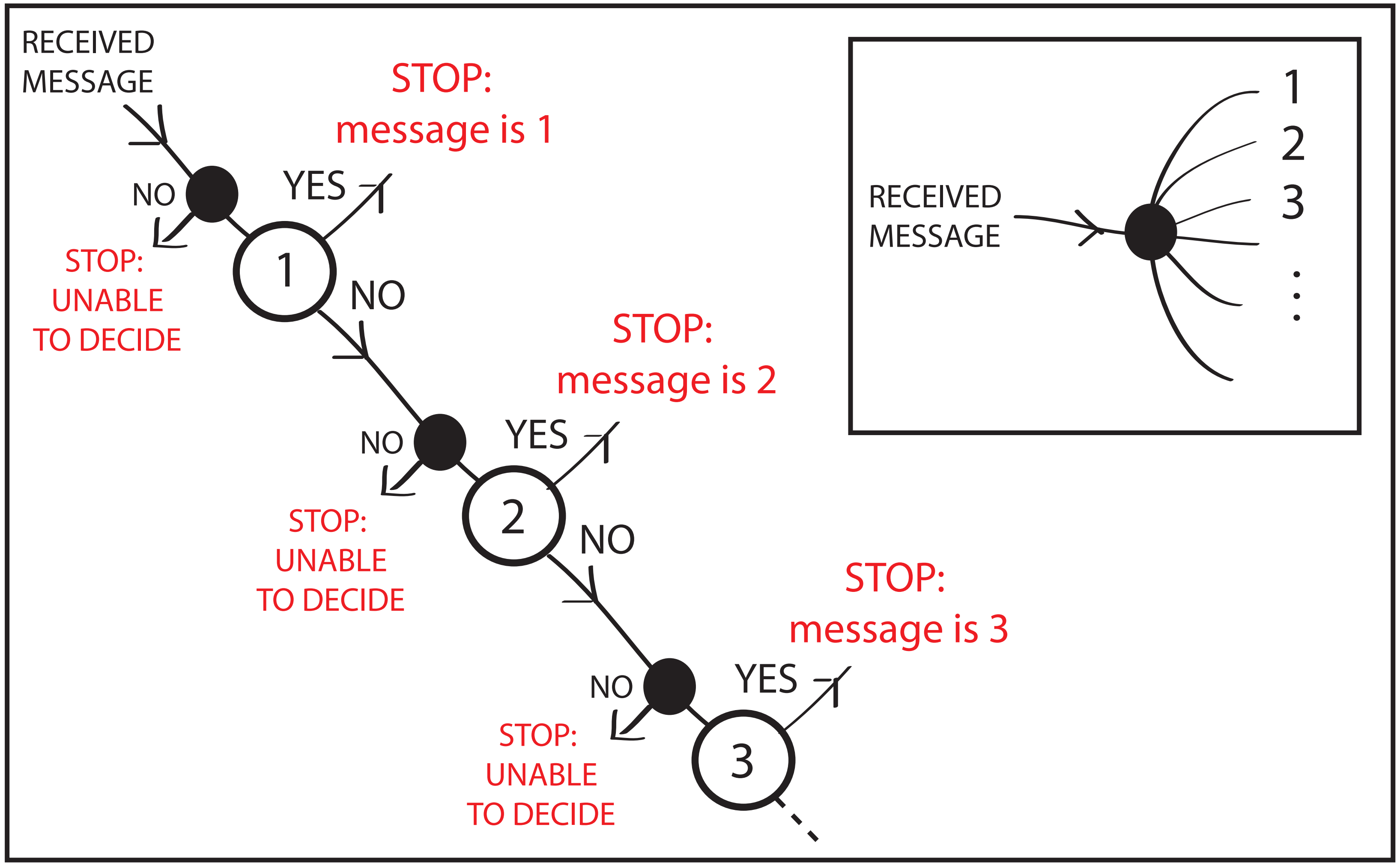
HOW MUCH INFO CAN BOB GAIN ON X ?

use channel capacity, i.e. the asymptotic optimal transmission rate

$$R = \frac{\log M}{N}$$

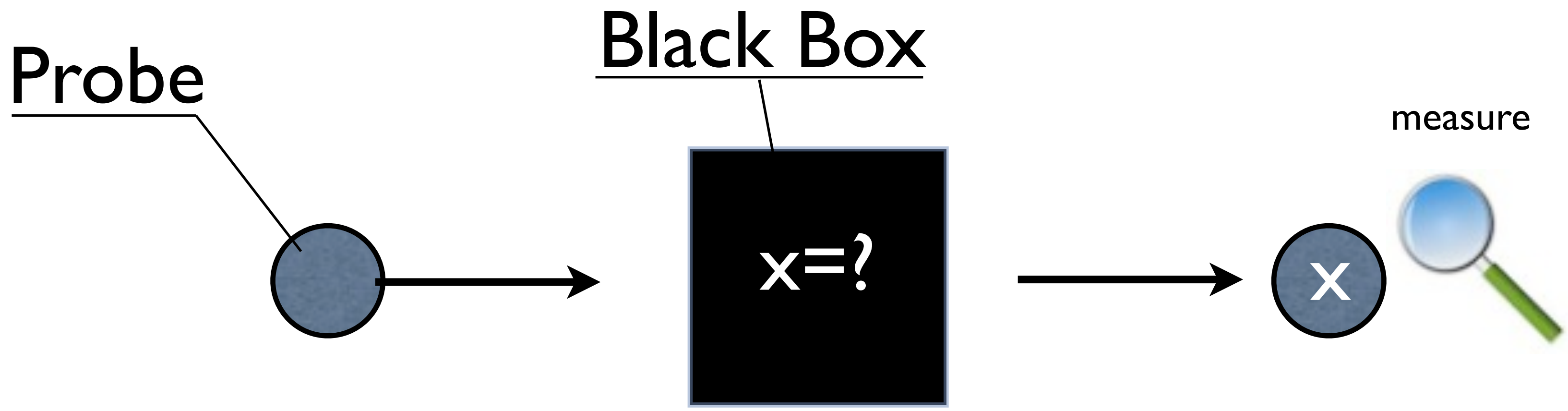


The optimal communication strategy can be constructed by focusing on Typical Subspaces; the associated POVM can then be implemented via Projective Measurements

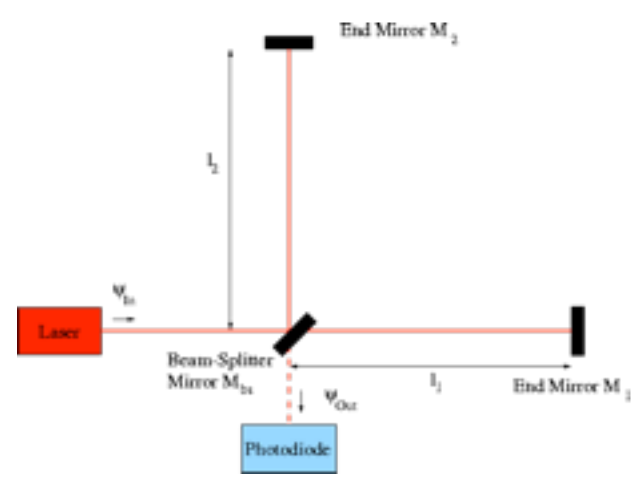


Lloyd, Giovannetti, Maccone PRL (2011)

Application V: Quantum Estimation



it transforms the
incoming probe
“encoding” on it
the unknown value x
of the parameter

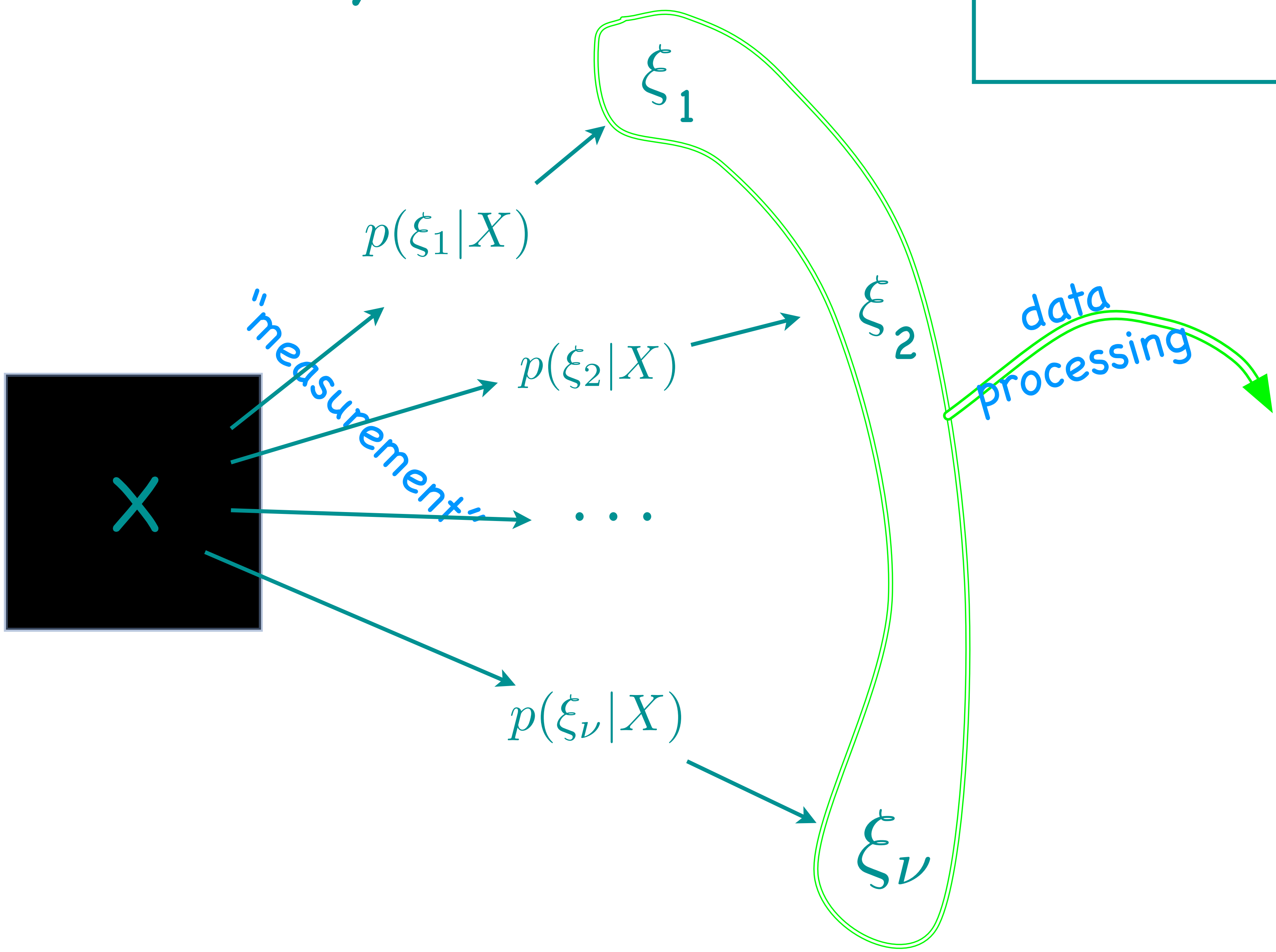


The physical mechanisms responsible of the encoding is known. The value of x is however unknown.

GOAL: determine X

Classical theory

$p(\xi|X)$ conditional probability of getting ξ given the value X



$\vec{\xi} \equiv (\xi_1, \xi_2, \cdots, \xi_\nu)$

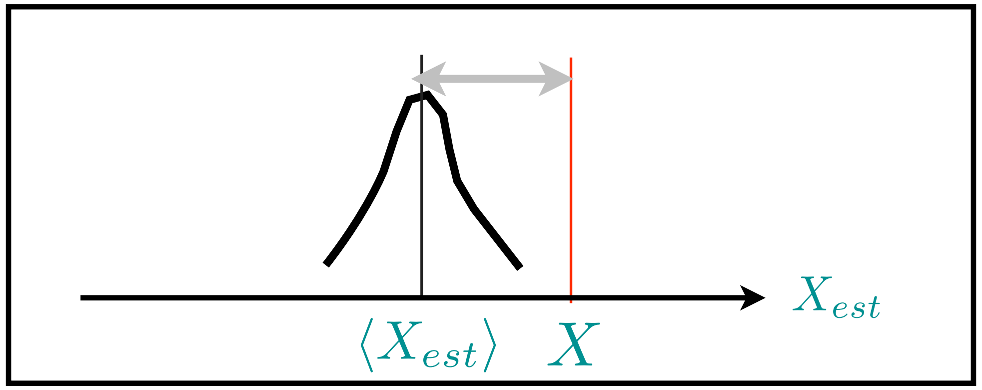
$$X_{est}(\vec{\xi})$$

Estimation of X after ν measurements

i.i.d. process $\quad P(\vec{\xi}|X) = \prod_{j=1}^{\nu} p(\xi_j|X)$

Root Mean Square Error (RMSE)

$$\delta X \equiv \sqrt{\sum_{\vec{\xi}} P(\vec{\xi}|X)\left[X_{\rm est}(\vec{\xi}) - X\right]^2} = \sqrt{\Delta X^2 + (X - \langle X_{est}\rangle)^2}$$



CRAMER-RAO bound

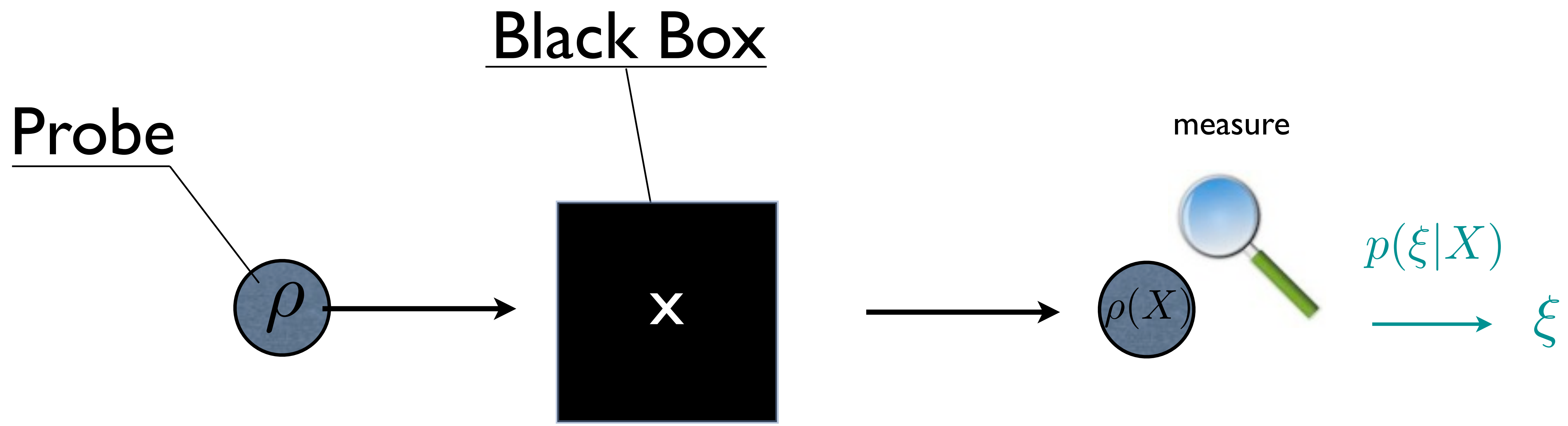
(for asymptotically unbiased est. strategies)

$$\delta X \geqslant \frac{1}{\sqrt{\nu F(X)}}$$

ACHIEVABLE FOR LARGE ENOUGH ν

$$F(X) \equiv \langle \Big[\frac{\partial}{\partial X} \ln p(\xi|X) \Big]^2 \rangle$$ FISHER information

$1/\sqrt{\nu}$ scaling with respect to the number times we repeat the measurement



POVM (positive operator valued measure)

$$\sum_{\xi} E_{\xi} = \mathbb{1} \qquad E_{\xi} \geqslant 0$$

$$p(\xi|X) = \mathrm{Tr}\left[E_{\xi}\ \rho(X)\right]$$

The physical mechanism which is responsible for the process is known. What we do NOT know is the value of the parameter X.

$$\rho \longrightarrow \rho(X) = \mathcal{E}_X(\rho)$$

Given $\rho(X)$, which is the best estimation of X we can get?

For each POVM we can write $$\delta X_{POVM} \geqslant \frac{1}{\sqrt{\nu F_{POVM}(X)}}$$

Therefore the optimal estimation error is given by

$$\boxed{\delta X \geqslant \frac{1}{\sqrt{\nu F_0(X)}}}$$

Q-CRAMER-RAO BOUND

Helstrom (1976)

where

$$F_0(X) \equiv \max_{POVM} F_{POVM}(X)$$

Q-FISHER INFO

maximum with respect to all POVM

$$\delta X \geqslant \frac{1}{\sqrt{\nu F_0(X)}} \geqslant \frac{1}{2\sqrt{\nu \Delta^2 H}}$$

$$\delta X \, \Delta H \geqslant \frac{1}{2\sqrt{\nu}}$$

Generalized "Energy-Time" uncertainty relation

Braunstein, Caves, Milburn, Ann Phys. (1996)

Q-CRAMER-RAO BOUND $\xrightarrow[\text{Prior info}]{}$ Quantum Ziv-Zakai BOUND

Tsang Phys. Rev. Lett. (2012)

Summary

- Basics Definitions
  - Projective Measurements
  - Indirect & noisy measurements POVM

- Applications to
  - State and Process Tomography
  - State and Process Discrimination
  - Quantum Communication
  - Quantum Estimation

NOT DISCUSSED:

-measurement in continuous-time
-waveform detection
-quantum back-action
- WAY theorem
-weak measurements
....

Extra Slides

DISTANCES BETWEEN STATES

Given two states of S it is important to define a "distance" among them.

TRACE DISTANCE

$$D(\rho_1, \rho_2) = \frac{1}{2}\mathrm{Tr}|\rho_1 - \rho_2| \qquad |\Theta| = \sqrt{\Theta^\dagger \Theta}$$

it is a "real distance" as it is
(i) symmetric,
(ii) positive semi-definite and nullifies iff the two states coincides,
(iii) satisfies the triangular inequality.

FIDELITY

$$F(\rho_1, \rho_2) = \mathrm{Tr}\left[\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right]$$

it is NOT a distance, but it has some good properties (~opposite of a distance)
(i) symmetric,
(ii) positive semi-definite and smaller than one
(it reaches one iff the two states coincides),
(iii) it can be used to bound D.

$$D(\rho_1, \rho_2) + F(\rho_1, \rho_2) \geqslant 1 \qquad D^2(\rho_1, \rho_2) + F^2(\rho_1, \rho_2) \leqslant 1$$

The most general time-evolution of a quantum system is described by assigning a map (channel) Φ which connects the input state $\rho \in \mathfrak{S}(\mathcal{H})$ of the system to its output counterpart $\rho' \in \mathfrak{S}(\mathcal{H})$

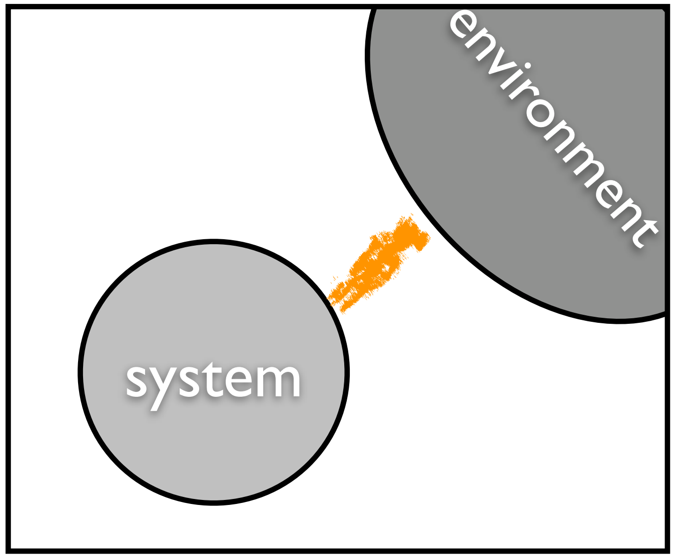
Such transformations must be LINEAR in the larger space of the operator algebra, COMPLETELY POSITIVE, and TRACE-PRESERVING (CPT).

output state

input state

$$\rho' = \Phi(\rho) = \mathrm{Tr}_E\left[U\ (\rho \otimes \sigma_E)\ U^\dagger\right]$$

they can always be represented as an UNITARY interaction with an external (possibly fictitious) environment.



STATES

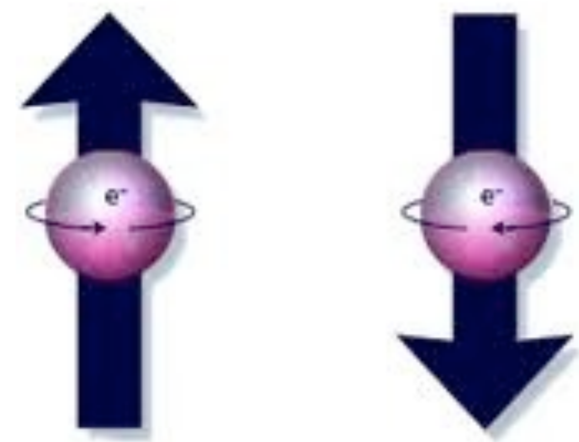
Example: the QUBIT ($\dim \mathcal{H} = d = 2$)

Example: *two level atom*

$|1\rangle = |e\rangle$

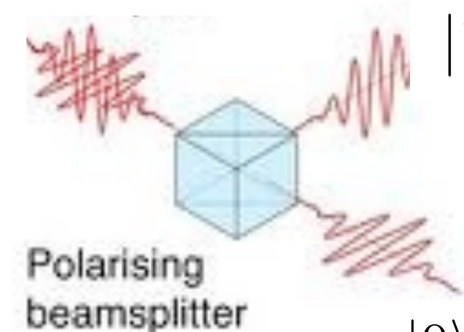
$|0\rangle = |g\rangle$

Example: *spin 1/2*



$|1\rangle = |\uparrow\rangle$ $\quad$ $|0\rangle = |\downarrow\rangle$

Example: *polarized single photon*



$|1\rangle = |\updownarrow\rangle$

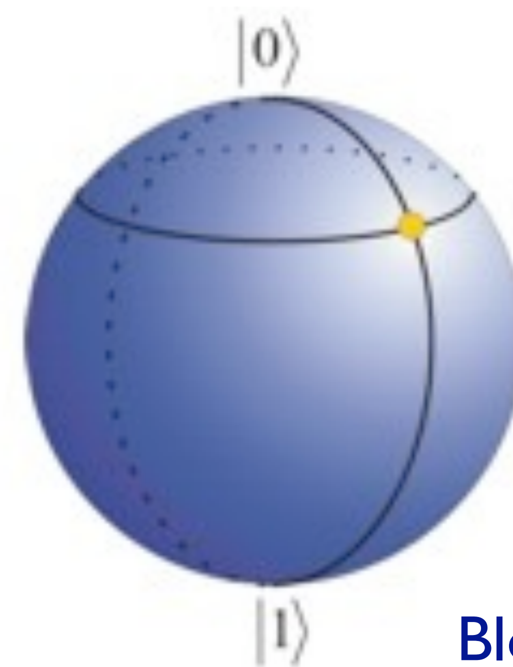
$|0\rangle = |\leftrightarrow\rangle$

$|0\rangle, |1\rangle$ $\qquad$ $\langle 1|0\rangle = 0$

superposition

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$



Bloch Sphere